EXERCISES: BASICS OF CENTRAL BANKS & MONETARY POLICY - 01

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Question 1. Starting from the dynamic intertemporal budget constraint for a joint fiscal-monetary authority,

$$dH_{CB,t} = H_{CB,t} r_t dt + \frac{M_t}{P_t}(r_t + \pi_t)dt + (T_t - G_t)dt, \tag{1}$$

derive the corresponding static intertemporal budget constraint

$$H_{CB,t} + E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} T_u du + E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t}(r_u + \pi_u)\frac{M_u}{P_u} du = E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} G_u du, \tag{2}$$

where Λ is a stochastic discount factor process. Hint: Consider $d(\Lambda_t H_{CB,t})$ and assume that $\lim_{T\to\infty} E_t[\Lambda_T H_{CB,T}] = 0$.

Now derive the following alternative form of the static intertemporal budget constraint and provide intuition for each term.

$$W_{CB,t} + E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t}\frac{dM_u}{P_u} + E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} T_u du = E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} G_u du, \tag{3}$$

where

$$W_{CB,t} = H_{CB,t} + \frac{M_t}{P_t}. \tag{4}$$

Question 2. Time is continuous, $t \in [0, \infty) = \mathcal{T}$. There is no risk. There is a representative agent with preferences defined over consumption rate and work flow. Her date-t utility is given by

$$\int_t^\infty e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du, \tag{5}$$

where

$$C_t = \left(\int_{i\in[0,1]} C_t(i)^{\frac{1}{1-\epsilon}} di \right)^{1-\epsilon}, \tag{6}$$

and $C_t(i)$ is the household's rate of consumption for good i.

Differentiated goods are produced by a continuum of firms, $i \in [0, 1]$, where firm i's date-t output flow is $Y_t(i)$, where

$$Y_t(i) = A_t N_t(i), \tag{7}$$

and $N_t(i)$ is firm i's labor input flow and A_t is the level of technological progress, which is common across firms, and is given by

$$\frac{dA_t}{dt} = \mu A_t, \text{ given } A_0 \tag{8}$$

The nominal wage rate paid to the household is W_t.

The financial wealth of the household can be invested in a nominal bond, which pays off 1 USD at an instant from now. The second is a claim on future dividends paid by the firms, where the date-t real dividend is given by

$$D_t = \int_{i\in[0,1]} D_t(i), \tag{9}$$

where $D_t(i)$ is the date-t dividend flow paid by firm i:

$$D_t(i) = Y_t(i) - \frac{W_t}{P_t} N_t(i). \tag{10}$$

1. Work out the household's dynamic intertemporal budget constraint. Hence, derive the static date-t intertemporal budget constraint
$$H_t = \int_t^\infty \frac{\Lambda_u}{\Lambda_t}\left(C_u - \frac{W_u}{P_u} N_u\right) dt. \tag{11}$$
2. Use the static formulation of the household's optimization problem to show that
$$\frac{\Lambda_u}{\Lambda_t} = e^{-\delta(u-t)}\left(\frac{C_u}{C_t}\right)^{-1} \tag{12}$$
and
$$\frac{W_t}{P_t} = N_t^{\varphi} C_t \tag{13}$$
Show that the real risk-free rate is given by
$$r_t = \delta + \frac{dc_t}{dt}, \tag{14}$$
where $c_t = \ln C_t$.
3. By imposing market clearing for the composite consumption good, show that
$$\frac{W_t}{P_t A_t} = N_t^{1+\varphi} \tag{15}$$
and
$$\frac{W_t}{A_t P_t} = \left(\frac{C_t}{A_t}\right)^{1+\varphi} \tag{16}$$
4. Using conditions arising from market clearing, show that
$$H_t + C_t \int_t^\infty e^{-\delta(u-t)}\left(\frac{C_u}{A_u}\right)^{1+\varphi} dt = \frac{1}{\delta} C_t. \tag{17}$$
How would you interpret the term $\int_t^\infty e^{-\delta(u-t)}\left(\frac{C_u}{A_u}\right)^{1+\varphi} dt$?

5. By considering the optimization problem of an individual firm, show that

$$\frac{W_t}{P_t A_t} = \frac{\epsilon - 1}{\epsilon}. \tag{18}$$

Hence show that

$$N_t = N = \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{1+\varphi}}. \tag{19}$$

6. Show that the real risk-free rate is given by

$$r_t = \delta + \frac{da_t}{dt} = \delta + \mu. \tag{20}$$

7. Prove that

$$\int_t^\infty e^{-\delta(u-t)} \left(\frac{C_u}{A_u}\right)^{1+\varphi} dt = \frac{1}{\delta} N. \tag{21}$$

Hence show that

$$H_t = \frac{1}{\epsilon\delta} C_t. \tag{22}$$

8. Prove directly that the date-t value of aggregate dividends, J_t, where

$$J_t = \int_t^\infty \frac{\Lambda_u}{\Lambda_t} D_u du, \tag{23}$$

is given by

$$J_t = \frac{D_t}{\delta}, \tag{24}$$

where

$$D_t = \frac{A_t N}{\epsilon} \tag{25}$$

9. What fraction of aggregate wealth is human capital?
10. Prove that

$$V_t = \sup_{(C_u)_{u\in\mathcal{T}},(N_u)_{u\in\mathcal{T}}} \int_t^\infty e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi}\right) du = \frac{a_t}{\delta} + \frac{\mu}{\delta^2} + \frac{1}{\delta}\left(\ln N - \frac{N^{1+\varphi}}{1+\varphi}\right), \tag{26}$$

where $a_t = \ln A_t$. Hence show that

$$V_t = \frac{\ln H_t + \ln(\epsilon\delta)}{\delta} + \frac{\mu}{\delta^2} - \frac{1}{\delta}\frac{N^{1+\varphi}}{1+\varphi}. \tag{27}$$

11. Show that

$$\ln N^n - \frac{(N^n)^{1+\varphi}}{1+\varphi} = \frac{1}{1+\varphi}\left[\ln\left(1 - \frac{1}{\epsilon}\right) - \left(1 - \frac{1}{\epsilon}\right)\right] \tag{28}$$

Question 3. Time is continuous, $t \in [0, \infty) = \mathcal{T}$. There is no risk. There is a representative agent with preferences defined over consumption rate and work flow. Her date-t utility is given by

$$\int_t^\infty e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du, \tag{29}$$

where

$$C_t = \left(\int_{i \in [0,1]} C_t(i)^{\frac{1}{1-\epsilon}} di \right)^{1-\epsilon}, \tag{30}$$

and $C_t(i)$ is the household's rate of consumption for good i.

Differentiated goods are produced by a continuum of firms, $i \in [0, 1]$, where firm i's date-t output flow is $Y_t(i)$, where

$$Y_t(i) = A_t N_t(i), \tag{31}$$

and $N_t(i)$ is firm i's labor input flow and A_t is the level of technological progress, which is common across firms, and is given by

$$\frac{dA_t}{dt} = \mu A_t, \text{ given } A_0 \tag{32}$$

The nominal wage rate paid to the household is W_t.

$D_t(i)$ is the date-t dividend flow paid by firm i:

$$D_t(i) = Y_t(i) - \frac{W_t}{P_t} N_t(i) - \Theta_t(i), \tag{33}$$

where $\Theta_t(i)$ is an adjustment cost function given by

$$\Theta_t(i) = \frac{1}{2}\theta \left(\frac{dP_t(i)/dt}{P_t(i)} \right)^2 P_t Y_t. \tag{34}$$

Firm i seeks to maximize the expected present value of its future dividends.

The financial wealth of the household can be invested in a nominal bond, which pays off 1 USD at an instant from now. The second is a claim on sum of future dividends paid by the firms plus their price adjustment costs, which is given by

$$\int_{i \in [0,1]} D_t(i) + \Theta_t(i) di. \tag{35}$$

(1) By considering the optimization problem of an individual firm and assuming that the price of good i is locally deterministic, show that

$$0 = \frac{\epsilon - 1}{\theta} \left(\frac{\epsilon}{\epsilon - 1} \frac{W_t}{P_t A_t} - 1 \right) + \frac{d\pi_t}{dt} - \delta \pi_t, \tag{36}$$

where date-t inflation is given by

$$\pi_t dt = \frac{dP_t}{P_t}. \tag{37}$$

(2) By imposing market clearing, show that

$$0 = \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x_t} - 1\right) + \frac{d\pi_t}{dt} - \delta\pi_t, \tag{38}$$

where

$$x = \ln X = \ln\frac{Y}{Y^n}, \tag{39}$$

where Y^n is natural output flow, i.e. output flow in the economy with no price adjustment costs ($\theta = 0$)

(3) Show that in equilibrium

$$H_t + e^{x_t}Y_t^n\frac{\epsilon - 1}{\epsilon}\int_t^\infty e^{-\delta(u-t)}e^{(1+\varphi)x_u}du = \frac{e^{x_t}Y_t^n}{\delta} \tag{40}$$

(4) Show that the nominal interest rate is given by

$$i_t = r_t + \pi_t. \tag{41}$$

Show also that the real risk-free rate is given by

$$r_t = r_t^n + \frac{dx_t}{dt}, \tag{42}$$

where r^n is the natural rate of interest. Hence, show that

$$i_t = r_t^n + \pi_t + \frac{dx_t}{dt}. \tag{43}$$

(5) Prove that

$$\pi_t = \frac{\epsilon - 1}{\theta}\int_t^\infty e^{-\delta(u-t)}\left(e^{(1+\varphi)x_u} - 1\right)du, \tag{44}$$

if

$$\lim_{T\to\infty} e^{-\rho(T-t)}\pi_T = 0. \tag{45}$$

Hence, show that the date-t value of human capital is given by

$$\left(\frac{1}{\epsilon}\delta\theta\pi_t + 1 - \frac{1}{\epsilon}\right)\frac{e^{x_t}Y_t^n}{\delta} \tag{46}$$

(6) Prove that, in equilibrium, the household's value function

$$V_t = \sup_{(C_u)_{u\in\mathcal{T}},(N_u)_{u\in\mathcal{T}}}\int_t^\infty e^{-\delta(u-t)}\left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi}\right)du, \tag{47}$$

is given by

$$V_t = V_t^n + \int_t^\infty e^{-\delta(u-t)}x_u du - \frac{1}{1+\varphi}\frac{\theta}{\epsilon}\pi_t, \tag{48}$$

where

$$V_t^n = \frac{a_t}{\delta} + \frac{\mu}{\delta^2} + \frac{1}{\delta}\left(\ln N^n - \frac{(N^n)^{1+\varphi}}{1+\varphi}\right), \tag{49}$$

and

$$N_t^n = N^n = \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{1+\varphi}}. \tag{50}$$