# Lecture 1: Basics of Central Banks & Monetary Policy

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# Overview

- What is money?
- Central Banks
- How do Central Banks conduct policy?
- Can we actually use policy to stabilize the economy?
- If so, which polices should we use?
- We shall focus on monetary policy and macroprudential policy (if we have time)

# Functions of Money I

- **Medium of exchange**. Money facilitates transactions as it is accepted by everybody in the exchange. This function is based on the perfect liquidity characteristic mentioned above. Obviously, the medium of exchange role of money has a tremendous impact on the volume of purchases and sales in the economy. Think for a moment of two economies with the same resources but one with money and one without money (barter economy). What would happen in the non-monetary economy? Problems such as the double coincidence of wants and non-divisible goods would arise making the exchanges very costly. Transaction costs would be much higher in this economy without money.

# Functions of Money II

- **Medium of account**. Another important function provided by money is serving as a medium of account that gives prices to all the distinct goods of the economy. This function of money also reduces substantially the transaction costs. In particular, the information costs are much lower in a monetary economy. The introduction of money allows us to express the value of all goods (their price) in terms of the units of money. In an economy with $N$ goods, there would be $N-1$ prices to learn. If this economy were a barter economy shoppers would have $N(N-1)/2$ relative prices to learn. As one illustrative example, I suggest you calculate the number of prices to memorize when there are 1000 goods in the economy ($N = 1000$).
- **Store of value** Income can be saved for the future in the form of money. As money does not deteriorate over time (at least over quite a long period of time!), it can be used to store value. In modern economies the store-of-value property is also found in other assets such as bonds, shares or real state properties. Since money pays no interest, it is considered inferior to any of the other store-of-value assets: bonds pay an interest yield, shares pay dividends and real state pay a rental rate. This inferiority is increased when

# Functions of Money III

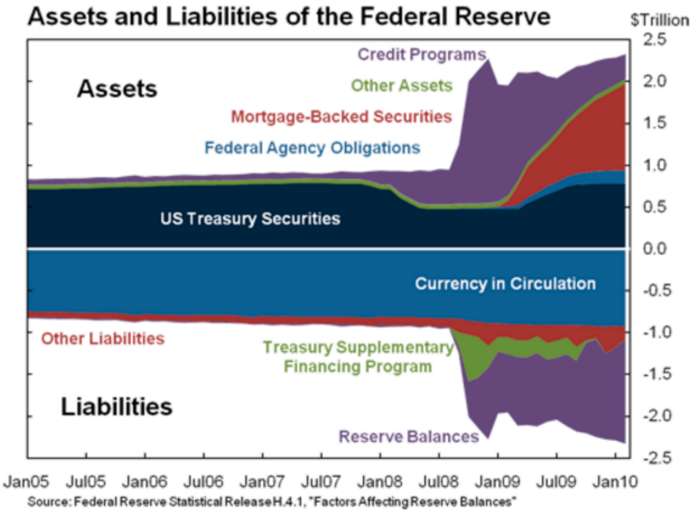
money is falling in value because of a positive rate of inflation. From a purely rational perspective, our savings should be placed in some risk-free interest-bearing asset and money demand for this purpose should be zero.

# Fiat Money

- Any money declared by a government to be legal tender.
- State-issued money which is neither convertible by law to any other thing, nor fixed in value in terms of any objective standard.
- Intrinsically valueless money used as money because of government decree.

Fiat money has been the norm since the 1970's. Potential problems as a store of value.

# Central Bank Balance Sheet I



# Central Bank Balance Sheet II

- Assets
  - Government tax revenues less subsidies are held at the Fed. Interest payments on government issued debt are made from this account. Basically, the Treasury has an account at the Fed
  - Assets owned by the Fed, such us Government Debt, MBS (Mortgage-backed securities).
- Liabilities
  - Reserve cash of US depository institutions (banks) – usually at least 10% of a bank's total deposits (this limit is called the **reserve requirement**).
  - US paper currency in circulation – when a US depository institution (bank) needs more currency because of increased demand for money from consumers, its asks for notes from the Fed. Upon receipt of the notes, the US depository institution's account at the Fed is debited. The liabilities on the Fed's balance sheet have been transformed from reserves to currency.
  - Government issued debt (because of the Treasury having an account at the Fed).
    - The Fed does not buy these securities directly from the Treasury. The Fed buys them from institutions, which have purchased US Government Debt

# Central Bank Balance Sheet III

- Suppose the Fed purchases $x$ USD worth of T-Bills from a bank. The bank receives an deposit of 10000 USD at its account at the Fed, which increases the Fed's liabilities by 10,000 USD and increases its assets via 10,000 USD worth of T-Bills. There is no net change in the value of the balance sheet, but both assets and liabilities have increased by the same amount.
- Any US Government Debt purchased by the Fed has been taken out of circulation, decreasing supply, thereby raising the price and lowering the relevant interest rate. Hence, the Government spends less cash to pay off its loans. By purchasing US Government Debt, a liability is cancelled out by an asset. The act of purchasing the debt via cash, creates a new form of liability. The initial liability in the form of the Treasury borrowing money gets transformed into cash and the relevant interest rate is lowered, saving the Government money. Furthermore, the Fed receives coupon payments on the US Government Debt it owns, which its gives back the to the Government! If the Fed holds the US Government Debt it has purchased, the Fed is allowing the US Government to borrow money for free.
- Look at Federal Reserve Bank of St. Louis, Is the Fed Monetizing Government Debt?, February 1, 2013

# Monetary Policy Tools

- Money supply
- Money demand – interest rate
- Quantitative Easing
- Credit Easing

# Money Supply I

- The old way to think about monetary policy was to consider changes in the size of the **money supply**. This is usually done through **open market operations**, in which **short-term government debt** is exchanged with the private sector.
- Remember that your standard US bank holds cash reserves at the Fed. The Fed requires that banks meet a reserve requirement (this called **fractional-reserve banking**). To ensure they meet this reserve requirement, banks borrow from each other overnight at a special interest rate, known as the **Fed funds rate** (in other countries, it's called the **interbank rate**. LIBOR is the UK name). The Fed funds rate floats depending on how much banks have to lend. The amount they borrow and lend each night is called Fed funds.

# Money Supply II

- If the Fed buys or borrows treasury bills from commercial banks, the central bank will add cash to the accounts, called **reserves**, that banks are required to keep with it. That expands the money supply. By contrast, if the Fed sells or lends treasury securities to banks, the payment it receives in exchange will reduce the money supply. The change in the amount of money in the economy affects **interbank interest rates**. Trading in short-term government debt affects the short-end of the yield curve.
- We can try and write down a **budget constraint** modelling this. We shall assume there is a joint fiscal-monetary authority, which can pay a subsidy at a rate $G_t$ to households and receive taxation revenues at the rate $T_t$. For simplicity, assume the only assets owned by this authority are short-term bonds of infinitesimal maturity, the stock of which is denoted in real terms by $W_{CB}$, which earns the real risk-free interest rate, $r$. The authority can also issue cash notes, which are liabilities. Denote the date-$t$ nominal stock of

# Money Supply III

money by $M$ and the price index by $P$. The authority's real liabilities are then $\frac{M}{P}$. Hence, the date-$t$ value of the fiscal-monetary authority's balance sheet is

$$H_{CB,t} = W_{CB,t} - \frac{M_t}{P_t}. \tag{1}$$

The date-$t + dt$ value of the balance sheet is given by

$$H_{CB,t+dt} = W_{B,t} e^{\int_t^{t+dt} r_u du} - \frac{M_t}{P_{t+dt}} + (T_t - G_t)dt \tag{2}$$

Assuming that the price-index is locally risk-free, we have

$$P_{t+dt} = P_t e^{\int_t^{t+dt} \pi_u}, \tag{3}$$

# Money Supply IV

where $\pi_t$ is date-$t$ inflation. Hence we obtain

$$H_{CB,t+dt} = W_{CB,t} e^{\int_t^{t+dt} r_u du} - \frac{M_t}{P_t} e^{-\int_t^{t+dt} \pi_u} \tag{4}$$

$$H_{CB,t+dt} = W_{CB,t}(1 + r_t dt) - \frac{M_t}{P_t}(1 - \pi_t dt) + (T_t - G_t)dt \tag{5}$$

$$dH_{CB,t} = W_{CB,t} r_t dt + \frac{M_t}{P_t} \pi_t dt + (T_t - G_t)dt \tag{6}$$

When the fiscal-monetary authority has issued money, inflation erodes the real value of its liabilities, thereby increasing the real value of the balance sheet. Recalling that $H_{CB} = W_{CB} - \frac{M}{P}$, we obtain

$$dH_{CB,t} = H_{CB,t} r_t dt + \frac{M_t}{P_t}(r_t + \pi_t)dt + (T_t - G_t)dt \tag{7}$$

# Money Supply V

We can write the above dynamic intertemporal budget constraint as the following static intertemporal budget constraint

$$H_{CB,t} + E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} T_u du + E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} (r_u + \pi_u) \frac{M_u}{P_u} du = E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} G_u du, \quad (8)$$

where $\Lambda$ is a stochastic discount factor process.

Interpretation of fiscal-monetary authority's static intertemporal budget constraint

$E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} T_u du$ – $PV_t$ of tax revenues

$E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} (r_u + \pi_u) \frac{M_u}{P_u} du$ – $PV_t$ of money flows supplied to households. $M_u$ is the flow of money, $r_u + \pi_u$ is nominal 'wage rate' earned by the authority from supplying one unit of money per unit time, $\frac{r_u + \pi_u}{P_u}$ is real 'wage rate' earned by the authority from supplying one unit of money per unit time.

$E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} G_u du$ – $PV_t$ of flow of government subsidies.

# Money Demand

- **Money demand**. Nowadays we think about monetary policy in terms of interest rates, which of course impact money demand.
- Setting banking-system lending or interest rates (such as the US overnight bank lending rate, the **federal funds discount Rate**, and the London Interbank Offer Rate, or Libor) in order to manage money demand is a major tool used by central banks. Ordinarily, a central bank conducts monetary policy by raising or lowering its interest rate target for the interbank interest rate. If the nominal interest rate is at or very near zero, the central bank cannot lower it further. Such a situation, called a **liquidity trap**, can occur, for example, during deflation or when inflation is very low.
- The Fed uses open market operations to adjust the supply of reserve balances to keep the federal funds rate–the interest rate at which depository institutions lend reserve balances to other depository institutions overnight–around the target established by the FOMC (Federal Open Market Committee). Other major central banks operate in a similar fashion.
- That a target is announced publicly is key.

# Unconventional monetary policy at the zero bound I

- During the past two years, central banks worldwide have cut policy rates sharplyin some cases to zeroexhausting the potential for cuts. Nonetheless, they have found unconventional ways to continue easing policy. One approach has been to purchase large quantities of financial instruments from the market. This so-called **quantitative easing** increases the size of the central banks balance sheet and injects new cash into the economy. Banks get additional reserves (the deposits they maintain at the central bank) and the money supply grows. A closely related option, **credit easing** may also expand the size of the central banks balance sheet, but the focus is more on the composition of that balance sheet–that is, the types of assets acquired. In the current crisis, many specific credit markets became blocked, and the result was that the interest rate channel did not work. Central banks responded by targeting those problem markets directly. For instance, the Fed set up a special facility to buy commercial paper (very short-term corporate debt) to ensure that businesses had continued access to working capital. It also bought mortgage-backed securities to sustain housing finance.

# Unconventional monetary policy at the zero bound II

- Some argue that credit easing moves monetary policy too close to industrial policy, with the central bank ensuring the flow of finance to particular parts of the market. But quantitative easing is no less controversial. It entails purchasing a more neutral asset like government debt, but it moves the central bank toward financing the governments fiscal deficit, possibly calling its independence into question. Now that the global economy appears to be recovering, the main concern has shifted to charting an exit strategy: how can central banks unwind their extraordinary interventions and tighten policy, to ensure that inflation does not become a problem down the road?
- Read The Rise and (Eventual) Fall in the Fed's Balance Sheet, January 2014, Federal Reserve Bank of St. Louis
- For liquidity traps, read: Managing a Liquidity Trap: Monetary and Fiscal Policy, Werning (2012) and The New-Keynesian Liquidity Trap, Cochrane (2015).

# Basic New-Keynesian Model I

- Assume perfect foresight (will introduce risk later)
- Consider an **RBC model with a friction** (sticky prices or costly price adjustment) which distorts allocations.
  - Log output flow in the frictionless economy is the **natural rate of output**. Real risk-free rate in the frictionless economy is called the **natural interest rate**, denoted by $r^n$
  - Difference between log output flow and the natural rate of output is called the output gap, denoted by $x$
- Monetary policy can offset the friction and potentially improve allocations. The friction creates a role for policy. Without frictions, policy is irrelevant or, even worse, policy can be a friction and distort allocations!
  - **rate of inflation** is denoted by $\pi$

# Basic New-Keynesian Model II

- Standard 3-equation model

$$\delta\pi_t = \frac{d\pi_t}{dt} + \kappa x_t, \text{ inflation matters for output gap because of pricing friction} \\ \text{– New Keynesian Philips Curve} \quad (9)$$

$$\frac{dx_t}{dt} = \psi(i_t - r_t^n - \pi_t), \text{ dynamic investment-savings equation} \quad (10)$$

$$i_t = v(x_t, \pi_t), \text{ rule for nominal interest rate} \quad (11)$$

- Provides simplified framework for thinking about monetary policy
- What do $x$ and $\pi$ look like for different nominal interest rate rules?
- What are the economic foundations for the 3 equations summarizing the model?

# Outline of Basic New Keynesian Model

- Assume perfect foresight
- Representative household is a consumer-worker
- Continuum of firms producing differentiated goods using labor. Firms exploit monopoly power – set prices to maximize firm value
- Can look at employment
- No real investment
- Growth is exogenous

Household's optimisation probem can be attacked using tools from Back (2010)
Need some tools from deterministic control theory to attack firm's problem.

# Typical Deterministic Optimal Control Problem

- $t \in \mathcal{T} = [0, \infty)$
- We have a 1-d state[1], $s$, which evolves over time according to the following law of motion

$$\frac{ds(t)}{dt} = f(s(t), c(t)) \tag{12}$$

- The starting value of the state is given by $s(0) = s_0$. The future values of the state will depend on the control variable $c$, which is also 1-d.
- An agent chooses the path of the control, $c(t)_{t \in \mathcal{T}}$. Her objective is to maximize the discounted value of some flow function. At time-$t$, the flow function is given by

$$u(s(t), c(t)) \tag{13}$$

- With a constant discount rate $\rho$, the agent's objective is given by

$$J(s_0) = \sup_{c(t)_{t \in \mathcal{T}}} \int_0^{\infty} e^{-\rho t} u(s(t), c(t)) dt \tag{14}$$

- What path should the agent choose?

[1] later on we shall deal with multidimensional states

# Hamiltonians

One way of finding the optimal control is to define a Hamiltonian and apply Pontryagin's Maximum Principle.

- The Hamiltonian is defined by

$$\mathcal{H}(s(t), c(t), \lambda(t)) = u(s(t), c(t)) + \lambda(t) f(s(t), c(t)). \tag{15}$$

- We can interpret $\mathcal{H}(s(t), c(t), \lambda(t))dt$ as the sum of the flow function to the agent over the next infinitesimal instant plus the present value of future inflows
  - $f(s(t), c(t))dt$ gives the increase in the state and $\lambda(t)$ converts this into the same units as the flow function
  - $\lambda(t)$ is called the **co-state** variable
    - economic interpretation – shadow price of the state variable – later will see that

$$\lambda(t) = \frac{\partial J(t)}{\partial s_t}, \tag{16}$$

where

$$J(t) = J(s(t)) = \sup_{c(u)_{t \geq u}} \int_t^\infty e^{-\rho(u-t)} u(s(u), c(u)) du \tag{17}$$

    - finance interpretation – $e^{-\rho t}\lambda(t)$ – the discount factor for converting date-$t$ values into date-0 values

# Pontryagin's Maximum Principle.

NSC conditions for a maximum:

$$\mathcal{H}_s(s(t), c(t), \lambda(t)) + \frac{d\lambda(t)}{dt} - \rho\lambda(t) = 0 \tag{18}$$

$$\mathcal{H}_c(s(t), c(t), \lambda(t)) = 0 \tag{19}$$

$$\frac{ds(t)}{dt} = f(s(t), c(t)) \tag{20}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)s(t) = 0 \tag{21}$$

- Second equation gives FOC for the control – can therefore write control in terms of state and co-state variables
- First and third equations give a coupled system of ode's for state and co-state variables, where time is the independent variable
- Boundary conditions
    - initial condition for the state $s(0) = s_0$
    - transversality condition, $\lim_{t\to\infty} e^{-\rho t}\lambda(t)s(t) = 0$
- $\lambda(0)$ has to be found!

# Where does Pontryagin's Maximum Principle come from?

- We can derive Pontryagin's Maximum Principle based on a variational argument
  - We shall prove necessity, but not sufficiency
- To understand the ideas behind variational argument, we shall briefly review
  - optimization in finite dimensional vector spaces, in particular how to maximize a functional with and without constraints and the idea of directional derivatives
  - redo the above with function spaces, a type of infinite dimensional vector space

# Functionals and Lagrange Multipliers

- We need to remind ourselves (or learn quickly!) about two things:
  - Maximizing functionals
  - Maximizing functionals subject to dynamic constraints
- To understand basic ideas, it's best to go back to basics – in this case, optimization in finite dimensional (fd) vector spaces

# Maximizing Functionals I

**Definition 1**

A functional is a map $V \to \mathbb{R}$, where $V$ is a vector space

- Suppose you are dot that can move around a 3-d world. This world is entirely flat, apart from 1 hill, which is the top half of the 3-d unit sphere. What is the co-ordinate of the highest point you can reach and how high is it? Without doing any maths, you know that the maximum height is 1 unit and the corresponding co-ordinate is $(x, y, z) = (0, 0, 1)$. Clearly, this problem has a unique solution.
- Now let's attack the problem mathematically. The height of a point in our 3-d world is given by

$$f(x, y) = \begin{cases} \sqrt{1 - (x^2 + y^2)} & , x^2 + y^2 \leq 1, \\ 0 & , x^2 + y^2 > 1 \end{cases} \tag{22}$$

- $f(x, y)$ is a functional, which maps elements of $\mathbb{R}^2$ to $\mathbb{R}$. It is an example of a functional.

# Maximizing Functionals II

- At the maximum, we have $f_x = f_y = 0$. We can write this more succinctly as

$$\underbrace{\nabla}_{=(\partial_x, \partial_y)} f = \underbrace{0}_{\text{zero vector}} \tag{23}$$

Doing the algebra

$$f_x = -\frac{x}{2}\left[1 - (x^2 + y^2)\right]^{-\frac{1}{2}} \tag{24}$$

$$f_y = -\frac{y}{2}\left[1 - (x^2 + y^2)\right]^{-\frac{1}{2}} \tag{25}$$

and so we see that $\nabla f = 0$ implies $x = y = 0$, where $f = 1$.

# Directional derivatives

Others ways of writing the FOC for an unconstrained optimum

- $\forall n \in \mathbb{R}^2 \ \frac{\partial f}{\partial n} = 0$

$$\frac{\partial f}{\partial n} = \nabla f \cdot n = \partial_x f \cdot n_x + \partial_y f \cdot n_y \tag{26}$$

- $\frac{\partial f}{\partial n}$ is the inner product of $\nabla f$ and $n$
- using different notation for the inner product

$$\frac{\partial f}{\partial n} =< \nabla f, n > \tag{27}$$

- introduce yet another notation (will make it easier to understand optimization in infinite dimensional spaces)

$$\delta(f, n) =< \nabla f, n > \tag{28}$$

- $\nabla f$ and $n$ are vectors but $\delta(f, n)$ is a scalar

# Maximizing Functionals & Function Spaces I

- We can also differentiate functionals where the underlying vector space is an infinite dimensional function space
- For $V = \{f \in C^\infty : \forall x \, g(x + 2\pi) = g(x)\}$

$$I_0[g] = \int_0^{2\pi} (g(x))^2 dx \tag{29}$$

$$I_1[g] = \int_0^{2\pi} (g(x))^2 + (g'(x))^2 dx \tag{30}$$

$$\tag{31}$$

- Generalize idea of directional derivative to function spaces

$$\delta(I, \phi) = \frac{d}{d\epsilon} I[g + \epsilon\phi]|_{\epsilon=0} \tag{32}$$

# Maximizing Functionals & Function Spaces II

- If we have functional $I[g]$, we want to investigate $I[g + \epsilon\phi]$, where $\phi(x)$ is the direction along which we take the derivative – it is a vector on our space, which makes it another function. This amounts to considering small variations made to $g$ and seeing what happens as $|\epsilon|$ grows from 0. If one of the gradients $\frac{d}{d\epsilon}I[g + \epsilon\phi]$ is not zero, then $g$ cannot be a local extremum for $I$.

# Calculus of Variations I

- We now use a variational argument to prove that the conditions in Pontryagin's Maximum Principle are indeed necessary for a maximum
- Set up the Lagrangian for the agent's control problem

$$L[s,c] = \int_0^\infty e^{-\rho t} u(s(t),c(t)) + \overbrace{e^{-\rho t}\lambda(t)}^{\text{Lagrange multiplier}} \left( f(s(t),c(t)) - \frac{ds(t)}{dt} \right) dt \qquad (33)$$

- Include the discount factor $e^{-\rho t}$ within the Lagrange multiplier – makes it easy to rewrite the Lagrangian in the following parsimonious form

$$L[s,c] = \int_0^\infty e^{-\rho t} \left[ u(s(t),c(t)) + \lambda(t) \left( f(s(t),c(t)) - \frac{ds(t)}{dt} \right) \right] dt \qquad (34)$$

- Think of the Lagrangian as a functional, which maps the paths of the state and the control into a scalar
- Path for the control is parameterized by the function $c(t) : \mathcal{T} \to \mathbb{R}$.
- The corresponding path for the state is given by the solution of the ode (56) subject to the initial condition $s(0) = s_0$, which is parameterized by the function $s(t) : \mathcal{T} \to \mathbb{R}$.

# Calculus of Variations II

- Suppose the agent changes the path for the control to $c'(t)_{t\in\mathcal{T}}$. Wlog,

$$c'(t) = c(t) + \epsilon p(t), \tag{35}$$

where $p(t)$ is some function $p(t) : \mathcal{T} \to \mathbb{R}$. The $\epsilon$ allows us to shrink the difference between the old and new paths for the control.

- The initial value of the state is fixed at $s_0$, but the remainder of the path for the state wull change – the new path for the state is denoted by $s'(t)_{t\in\mathcal{T}}$. Wlog,

$$s'(t) = s(t) + \epsilon q(t), \tag{36}$$

where $q(t)$ is some function $q(t) : \mathcal{T} \to \mathbb{R}$ and $q(0) = 0$ to ensure the new path for the state still starts at $s_0$

- With the new paths for the control and the state, the Lagrangian becomes

$$L[s+\epsilon q, c+\epsilon p] = \int_0^\infty e^{-\rho t}\left[u(s(t)+\epsilon q(t), c(t)+\epsilon p(t))\right. \tag{37}$$

$$\left. +\lambda(t)\left(f(s(t)+\epsilon q(t), c(t)+\epsilon p(t)) - \frac{ds(t)}{dt} - \epsilon\frac{dq(t)}{dt}\right)\right] dt \tag{38}$$

- The agent wants to choose the path for the control, which maximizes the Lagrangian.

# Calculus of Variations III

- The original path is optimal if the following FOC holds

$$\forall q, p \ \frac{dL[s+\epsilon q, c+\epsilon p]}{d\epsilon}|_{\epsilon=0} = 0 \tag{39}$$

- We are just using the directional derivative $\delta(L[s, c], q, p) = \frac{dL[s+\epsilon q, c+\epsilon p]}{d\epsilon}|_{\epsilon=0}$, so our FOC is equivalent to

$$\forall q, p \ \delta(L[s, c], q, p) = 0 \tag{40}$$

- Let's compute the directional derivative

$$\frac{dL[s+\epsilon q, c+\epsilon p]}{d\epsilon}|_{\epsilon=0} \tag{41}$$

$$= \int_0^\infty e^{-\rho t}\left[\frac{d}{d\epsilon}\left[u(s'(t), c'(t)) + \lambda(t) f(s'(t), c'(t))\right] - \lambda(t)\frac{dq(t)}{dt}\right] dt|_{\epsilon=0}. \tag{42}$$

- We now introduce the **Hamiltonian** function

$$\mathcal{H}(s(t), c(t), \lambda(t)) = u(s(t), c(t)) + \lambda(t) f(s(t), c(t)). \tag{43}$$

# Calculus of Variations IV

- We have a nice economic interpretation for the Hamiltonian – it also makes our computations cleaner!

$$\frac{dL[s+\epsilon q, c+\epsilon p]}{d\epsilon}|_{\epsilon=0} = \int_0^\infty e^{-\rho t}\left[\frac{d}{d\epsilon}\mathcal{H}(s'(t), c'(t), \lambda(t)) - \lambda(t)\frac{dq(t)}{dt}\right] dt|_{\epsilon=0} \quad (44)$$

$$\frac{d}{d\epsilon}\mathcal{H}(s'(t), c'(t), \lambda(t)) = \mathcal{H}_s(s'(t), c'(t), \lambda(t))q(t) + \mathcal{H}_c(s'(t), c'(t), \lambda(t))p(t) \quad (45)$$

$$\frac{dL[s+\epsilon q, c+\epsilon p]}{d\epsilon}|_{\epsilon=0} \quad (46)$$

$$= \int_0^\infty e^{-\rho t}\left[\frac{d}{d\epsilon}\mathcal{H}(s'(t), c'(t), \lambda(t)) - \lambda(t)\frac{dq(t)}{dt}\right] dt|_{\epsilon=0} \quad (47)$$

$$= \int_0^\infty e^{-\rho t}\left[\mathcal{H}_s(s(t), c(t), \lambda(t))q(t) + \mathcal{H}_c(s(t), c(t), \lambda(t))p(t) - \lambda(t)\frac{dq(t)}{dt}\right] dt \quad (48)$$

# Calculus of Variations V

$$\int_0^\infty e^{-\rho t}\lambda(t)\frac{dq(t)}{dt}dt = \left[e^{-\rho t}\lambda(t)q(t)\right]_0^\infty - \int_0^\infty \frac{d}{dt}\left[e^{-\rho t}\lambda(t)\right]q(t)dt \tag{49}$$

$$= \lim_{t\to\infty} e^{-\rho t}\lambda(t)q(t) - \int_0^\infty e^{-\rho t}\left(\frac{d\lambda(t)}{dt} - \rho\lambda(t)\right)q(t)dt \tag{50}$$

The FOC then becomes

$$\int_0^\infty e^{-\rho t}\left[\left(\mathcal{H}_s(s(t),c(t),\lambda(t)) + \frac{d\lambda(t)}{dt} - \rho\lambda(t)\right)q(t) + \mathcal{H}_c(s(t),c(t),\lambda(t))p(t)\right] \tag{51}$$

$$- \lim_{t\to\infty} e^{-\rho t}\lambda(t)q(t),\ \forall p,q \tag{52}$$

Hence

$$\mathcal{H}_s(s(t),c(t),\lambda(t)) + \frac{d\lambda(t)}{dt} - \rho\lambda(t) = 0 \tag{53}$$

$$\mathcal{H}_c(s(t),c(t),\lambda(t)) = 0 \tag{54}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)q(t) = 0 \tag{55}$$

The latter equation holds for any state variable $q$ and so $\lim_{t\to\infty} e^{-\rho t}\lambda(t)s(t) = 0$

# Hamilton-Jacobi-Bellman Equation

Another approach to finding the optimal control

- $t \in \mathcal{T} = [0, \infty)$
- We have a 1-d state[2], $s$, which evolves over time according to the following law of motion

$$\frac{ds(t)}{dt} = f(s(t), c(t)) \tag{56}$$

- The starting value of the state is given by $s(0) = s_0$. The future values of the state will depend on the control variable $u$, which is also 1-d.
- An agent chooses the path of the control, $c(t)_{t \in \mathcal{T}}$. Her objective is to maximize the discounted value of some flow function. At time-$t$, the flow function is given by

$$u(s(t), c(t)) \tag{57}$$

- With a constant discount rate $\rho$, the agent's objective is given by

$$J(0) = J(s(0)) = J(s_0) = \sup_{c(t)_{t \in \mathcal{T}}} \int_0^{\infty} e^{-\rho t} u(s(t), c(t)) dt \tag{58}$$

- Date-$t$ objective function

$$J(t) = J(s(t)) = \sup_{c(u)_{t \geq u}} \int_t^{\infty} e^{-\rho(u-t)} u(s(u), c(u)) du \tag{59}$$

- The maximized objective function is called the **value function**
- What path should the agent choose?

[2]later on we shall deal with multidimensional states

The value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{c(t)} u(s(t), c(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c(t)) \tag{60}$$

- The FOC condition of the HJB is

$$u_c(s(t), c(t)) = J'(s(t)) f_c(s(t), c(t)) \tag{61}$$

- Solving the above equation gives the date-$t$ value of the optimal control in terms of the date-$t$ state and the derivative of the value function.
- To find the value function, substitute the optimal control, $c^*(t)$ into the HJB to get a nonlinear ordinary differential equation (no longer need the sup)

$$0 = u(s(t), c^*(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c^*(t)) \tag{62}$$

- How do we solve the above nonlinear ode
  - Does not generally have a closed-form solution – need numerical methods
  - Do not have boundary conditions – need a new concept of what a solution to a differential equation is

# Bellman's Principle of Optimality

Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)

Key conceptual difference relative to Pontryagin's Maximum Principle

- Think of the control a function of the state.

# Heuristic derivation of HJB I

Split the integral for the value function between the next small time interval, $[t, t+\Delta t)$ and the rest of time $[t+\Delta t, \infty)$

$$J(s(t)) = \sup_{c(u)_{u \geq t}} \int_t^{t+\Delta t} e^{-\rho(u-t)} u(s(u), c(u)) du + e^{-\rho \Delta t} \int_{t+\Delta t}^{\infty} e^{-\rho(u-[t+\Delta t])} u(s(u), c(u)) du \tag{63}$$

Apply Bellman's Principle of Optimality

- Suppose we choose an optimal path at date-$t$: $(c_u^t(s))_{u \geq t}$
- Suppose we choose an optimal path at date-$\tau > t$: $(c_u^\tau(s))_{u \geq \tau}$
- Principle of Optimality $\Rightarrow$ $(c_u^t(s))_{u \geq t} = (c_u^t(s))_{t \leq u < \tau} \cup (c_u^\tau)_{u \geq \tau}$

If we choose an optimal path for the control at some future date, when considered as functions of the state, the future optimal path is contained within today's. In other words, the choice of optimal control is time consistent.

For our infinite horizon problem, where the law of motion for the state does not depend explicitly on time, we can go even further: thinking of the control as a map from the state to the reals, it so happens that the map is time invariant, i.e. $\forall t \in \mathcal{T}$, $(c_u^t(s))_{u \geq t}$, for each $u \geq t$, we have $c_u^t(s) = c(s)$

# Heuristic derivation of HJB II

Exploiting the Principle of Optimality reveals a recursive structure for the value function

$$J(s(t)) = \sup_{c(u)_{u\in[t,t+\Delta t)}} \int_t^{t+\Delta t} e^{-\rho(u-t)} u(s(u),c(u))du + e^{-\rho\Delta t} \underbrace{\sup_{c(u)_{u\geq t+\Delta t}} \int_{t+\Delta t}^{\infty} e^{-\rho(u-[t+\Delta t])} u(s(u),c(u))du}_{J(s(t+\Delta t))} \tag{64}$$

To derive the HJB equation we just need to indulge in some Calculus. First observe that

$$\int_t^{t+\Delta t} e^{-\rho(u-t)} u(s(u),c(u))du = u(s(t),c(t))\Delta t + o(\Delta t) \tag{65}$$

$\left[h(t) = o(j(t)) \iff \lim_{t\to 0} \frac{h(t)}{j(t)} = 0\right]$ From the Principle of Optimality

$$\sup_{c(u)_{u\in[t,t+\Delta t)}} \int_t^{t+\Delta t} e^{-\rho(u-t)} u(s(u),c(u))du = \sup_{c(t)} u(s(t),c(t))\Delta t + o(\Delta t) \tag{66}$$

# Heuristic derivation of HJB III

Hence

$$J(s(t)) = \sup_{c(t)} u(s(t), c(t))\Delta t + e^{-\rho \Delta t} J(s(t + \Delta t)) + o(\Delta t) \tag{67}$$

Now

$$e^{-\rho \Delta t} = 1 - \rho \Delta t + o(\Delta t), \tag{68}$$

and so

$$J(s(t)) = \sup_{c(t)} u(s(t), c(t))\Delta t + J(s(t + \Delta t)) - \rho \Delta t J(s(t + \Delta t)) + o(\Delta t) \tag{69}$$

Furthermore $\Delta t J(s(t + \Delta t)) = \Delta t J(s(t)) + o(\Delta t)$, and so

$$J(s(t)) = \sup_{c(t)} u(s(t), c(t))\Delta t + J(s(t + \Delta t)) - \rho \Delta t J(s(t)) + o(\Delta t) \tag{70}$$

Now

$$J(s(t + \Delta t)) = J(s(t)) + \Delta t J'(s(t)) \frac{ds(t)}{dt} + o(\Delta t) \tag{71}$$

$$= J(s(t)) + \Delta t J'(s(t)) f(s(t), c(t)) + o(\Delta t) \tag{72}$$

(73)

# Heuristic derivation of HJB IV

and so

$$0 = \sup_{c(t)} \left( u(s(t), c(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c(t)) \right) \Delta t + o(\Delta t) \tag{74}$$

Dividing by $\Delta t$ and letting $\Delta t \to 0$ gives the HJB equation

$$0 = \sup_{c(t)} u(s(t), c(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c(t)) \tag{75}$$

# Connection between Pontrayagin's Maximum Principle and the Hamilton-Jacobi-Bellman Equation

- Write HJB equation in terms of Hamiltonian
- Identify $\lambda(t) = J'(s(t))$ (useful to observe that $\lambda(t) = \lambda(s(t))$)

$$0 = \sup_{c(t)} \mathcal{H}(s(t), c(t), J'(s(t))) - \rho J(s(t)) = 0 \tag{76}$$

- FOC of HJB gives us one part of Maximum Principle

$$\mathcal{H}_c(s(t), c(t), J'(s(t))) = 0 \tag{77}$$

- What about $\mathcal{H}_s(s(t), c(t), \lambda(t)) - \frac{d\lambda(t)}{dt} - \rho\lambda(t) = 0$?
  - We can also derive this from the HJB!

Start by noting that at the optimum, where $c(t) = c^*(s(t))$, the HJB becomes the following ode

$$0 = \mathcal{H}(s(t), c^*(s(t)), J'(s(t))) - \rho J(s(t)) \tag{78}$$

To derive $\mathcal{H}_s(s(t), c^*(t), \lambda(t)) - \frac{d\lambda(t)}{dt} - \rho\lambda(t) = 0$ simply differentiate the ode wrt to $s(t)$ and use the fact that $\mathcal{H}_c(s(t), c^*(s(t)), J'(s(t)))$

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), J'(s(t))) + \overbrace{\mathcal{H}_c(s(t), c^*(s(t)), \lambda(t))}^{=0} \frac{\partial c^*(s(t))}{s(t)} \tag{79}$$

$$+ \underbrace{\mathcal{H}_\lambda(s(t), c^*(s(t)), J'(s(t)))}_{=f(s(t), c^*(t))} J''(s(t)) - \rho J'(s(t)) \tag{80}$$

Remember the identification $\lambda(t) = J'(s(t))$, which implies $\frac{\partial \lambda(t)}{\partial s(t)} = J''(s(t))$

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), \lambda(t)) + f(s(t), c^*(t)) \frac{\partial \lambda(t)}{\partial s(t)} - \rho\lambda(t) \tag{81}$$

Noting that $\frac{d\lambda(t)}{dt} = \frac{\partial \lambda(t)}{\partial s(t)} \frac{ds(t)}{dt} = \frac{\partial \lambda(t)}{\partial s(t)} f(s(t), c^*(t))$, we obtain

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), \lambda(t)) + \frac{d\lambda(t)}{dt} - \rho\lambda(t) \tag{82}$$

**What is the point of having these two different approaches to a deterministic optimal control problem?** The Maximum Principle does not assume time consistency, whereas the

Hamilton-Jacobi-Bellman equation does. Ask yourself whether or not you think central banks or policy-making institutions are time consisent!
To learn more about optimal control, read Fleming & Rishel (1975) for rigorous theory and Chapter 7 of Acemoglu (2008) for economic applications.

# Model I

- Representative household

$$\int_t^{\infty} e^{-\rho(u-t)} \left( \ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du \tag{83}$$

  - $C$, consumption rate of composite good
  - $N$, rate of labor supply

- Continuum of firms $i \in [0,1]$ produces differentiated goods

$$Y_t(i) = A_t N_t(i) \tag{84}$$

  - $A_t$, common exogenous tech level

- Composite good defined by basket

$$C_t = \left( \int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{1}{1-\frac{1}{\epsilon}}} \tag{85}$$

# Model II

- $C_t(i)dt$ is the quantity of good $i$ consumed by the household over the interval $[t, t + dt)$.
- Can show that:
  - nominal expenditure on consumption aggregates nicely

$$P_t C_t = \int_0^1 P_t(i) C_t(i) di \tag{86}$$

where

$$P_t = \left[ \int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}} \tag{87}$$

$$\forall i \in [0, 1], \;\; C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t \tag{88}$$

- static intertemporal budget constraint, where $H$ is household wealth

$$H_0 = \int_0^\infty \Lambda_t \left( C_t - \frac{W_t}{P_t} N_t \right) dt \tag{89}$$

# Model III

- Lagrangian

$$\mathcal{L} = \int_0^\infty e^{-\rho t} \left( \ln C_t - \frac{N_t^{1+\varphi}}{1+\varphi} \right) dt - \kappa \int_0^\infty \Lambda_t \left( C_t - \frac{W_t}{P_t} N_t \right) dt \qquad (90)$$

- FOC's
  - consumption

$$e^{-\rho t} C_t^{-1} = \kappa \Lambda_t \qquad (91)$$

  - labor

$$e^{-\rho t} N_t^{\varphi} = \kappa \Lambda_t \frac{W_t}{P_t} \qquad (92)$$

- Implications of FOC's

# Model IV

- DF process

$$e^{-\rho(u-t)} \left( \frac{C_u}{C_t} \right)^{-1} = \frac{\Lambda_u}{\Lambda_t} \tag{93}$$

- consumption-labor

$$N_t^{\varphi} C_t = \frac{W_t}{P_t} \tag{94}$$

# Individual Firm's Deterministic Optimal Control Problem I

- Objective of firm $j$ is to set prices in order to maximize firm value net of adjustment costs

$$\sup_{dP_t(j)/dt} \int_t^{\infty} e^{-\int_t^u i_u} \left[\Pi_u(j) - \Theta_u(j)\right] du \tag{95}$$

  - real firm value is the above nominal value dividend by $P_t$, the aggregate price index

- Nominal profit flow function

$$\Pi_t(j) = P_t(j) Y_t(j) - W_t N_t(j) \tag{96}$$

# Individual Firm's Deterministic Optimal Control Problem II

- Adjustment cost function

$$\Theta_t = \frac{1}{2}\theta\left(\frac{dP_t(j)/dt}{P_t(j)}\right)^2 P_t Y_t \tag{97}$$

$$\Pi_t(j) = \left[\left(\frac{P_t(j)}{P_t}\right)^{1-\epsilon} - \frac{W_t}{A_t P_t}\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon}\right] P_t Y_t \tag{98}$$

- Simpler notation: $x_t = P_t(j)$, $\mu_t = dP_t(j)/dt$
- Deterministic optimal control problem

$$\sup_{\mu_t} \int_t^{\infty} e^{-\int_t^u i_u}\left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u}x_u^{-\epsilon}\right) P_u^{\epsilon} Y_u - \frac{1}{2}\theta\left(\frac{\mu_u}{x_u}\right)^2 P_u Y_u\right] du \tag{99}$$

$$\text{s.t. } dx_t = \mu_t dt \tag{100}$$

# Individual Firm's Deterministic Optimal Control Problem III

- $x$ is the state variable, which is the price of the good produced by the firm
- $\mu$, the rate of change of the state variable, is the control

- Hamiltonian

$$\mathcal{H} = \left( x_t^{1-\epsilon} - \frac{W_t}{A_t} x_t^{-\epsilon} \right) P_t^{\epsilon} Y_t - \frac{1}{2}\theta \left( \frac{\mu_t}{x_t} \right)^2 P_t Y_t + \mu_t \lambda_t \tag{101}$$

- Apply Pontryagin's Maximum Principle

$$(1-\epsilon)x^{-\epsilon} + \epsilon \frac{W}{A} x^{-(1+\epsilon)}] P^{\epsilon} Y + \theta \mu^2 x^{-3} P Y + \dot{\lambda} - i\lambda = 0 \tag{102}$$

$$\lambda = \theta \frac{\mu}{x^2} P Y \tag{103}$$

# Individual Firm's Deterministic Optimal Control Problem IV

- Symmetric equilibrium, where $x = P$. Define $\mu_P = \mu/P$

$$0 = \left[(1-\epsilon) + \epsilon \frac{W}{AP}\right] Y + \theta \mu_P^2 Y + \dot{\lambda} - i\lambda \tag{104}$$

$$\lambda = \theta \mu_P Y \tag{105}$$

- With no adjustment costs, i.e. $\theta = 0$, then

$$P_t = \frac{\epsilon}{\epsilon - 1} \frac{W_t}{A_t} \tag{106}$$

# Individual Firm's Deterministic Optimal Control Problem V

- Differentiate the second equation wrt $t$: $\dot{\lambda} = \theta \dot{\mu}_P Y + \mu_P \dot{Y}$ and use this together with the original second eqn to eliminate $\dot{\lambda}$ and $\lambda$ in the first equation. We thus obtain

$$\left( i - \mu_P - \frac{\dot{Y}}{Y} \right) \mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta} \left( \frac{\epsilon}{\epsilon - 1} \frac{W}{PA} - 1 \right) \tag{107}$$

# Bond Pricing I

- The real discount factor function is given by

$$\Lambda_t = e^{-\rho t} C_t^{-1} \tag{108}$$

  - Real price of a real bond paying off 1 unit of the composite consumption good at date $t + dt$

$$B_t = e^{-r_t dt} = \frac{\Lambda_{t+dt}}{\Lambda_t} \tag{109}$$

$$1 - r_t dt = 1 + \frac{d\Lambda_t}{\Lambda_t} \tag{110}$$

$$r_t = -\frac{\dot{\Lambda}_t}{\Lambda_t} = \rho + \frac{\dot{C}_t}{C_t} \tag{111}$$

  - define $c = \ln C$

$$r_t = \rho + \dot{c}_t \tag{112}$$

# Bond Pricing II

- Nominal discount factor function is given by

$$\Lambda_t^{\$} = e^{-\rho t} C_t^{-1} P_t^{-1} \tag{113}$$

  - nominal price of a nominal bond paying off 1 USD date $t + dt$

$$B_t^{\$} = e^{-i_t dt} = \frac{\Lambda_{t+dt}}{\Lambda_t} \frac{P_t}{P_{t+dt}} \tag{114}$$

$$1 - i_t dt = 1 + \frac{d\Lambda_t}{\Lambda_t} - \frac{dP_t}{P_t} + O(dt^2) \tag{115}$$

$$i_t = -\frac{\dot{\Lambda}_t}{\Lambda_t} + \frac{dP_t}{P_t} + O(dt) \tag{116}$$

  - In continuous time limit

$$i_t = -\frac{\dot{\Lambda}_t}{\Lambda_t} + \frac{dP_t}{P_t} \tag{117}$$

$$i_t = r_t + \mu_{P,t} \tag{118}$$

# Bond Pricing III

- Summary of bond pricing equations

$$i_t = r_t + \mu_{P,t} \tag{119}$$
$$r_t = \rho + \dot{c}_t \tag{120}$$
$$\tag{121}$$

# Equilibrium I

- $C = Y$

$$i = r + \mu_P \tag{122}$$

$$r = \rho + \dot{y} \tag{123}$$

$$(i - \mu_P - \dot{y})\,\mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta}\left(\frac{\epsilon}{\epsilon - 1}\frac{W}{PA} - 1\right) \tag{124}$$

- ode for $\mu_P$

$$\rho\mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta}\left(\frac{\epsilon}{\epsilon - 1}\frac{W}{PA} - 1\right) \tag{125}$$

- Finding $\frac{W}{P}$

# Equilibrium II

- Aggregate output, $Y_t = \int_0^1 Y_t(i)di$, Aggregate labor supply $N_t = \int_0^1 N_t(i)di$

$$Y_t = A_t \int_0^1 N_t(i) = A_t N_t \tag{126}$$

- Combine with FOC $N^{\varphi} Y = W/P$

$$N^{\varphi} AN = W/P \tag{127}$$

$$\tag{128}$$

- Obtain expression linking labor and wage rate

$$\frac{W}{PA} = N^{1+\varphi} \tag{129}$$

- Implies expression linking output and wage rate

$$\frac{W}{PA} = \left(\frac{Y}{A}\right)^{1+\varphi} = e^{(1+\varphi)(y-a)} \tag{130}$$

# Equilibrium III

- Summary of equilibrium conditions

$$i = r + \mu_P \tag{131}$$

$$r = \rho + \dot{y} \tag{132}$$

$$\left(i - \mu_P - \dot{y}\right)\mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta}\left[\frac{\epsilon}{\epsilon - 1}e^{(1+\varphi)(y-a)} - 1\right] \tag{133}$$

- Simplified equilibrium condition

$$\rho\mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta}\left[\frac{\epsilon}{\epsilon - 1}e^{(1+\varphi)(y-a)} - 1\right] \tag{134}$$

- Insight – can use $\mu_P$ and $\dot{\mu}_P$ to control output flow!

# Equilibrium IV

- Natural economy – limit with no price adjustment costs (multiply through by $\theta$ and let $\theta \to 0$)

$$\frac{\epsilon}{\epsilon - 1} e^{(1+\varphi)(y^n - a)} - 1 \tag{135}$$

- Can obtain equilibrium condition in terms of log output gap $x = y - y^n$

$$\rho \mu_P = \dot{\mu}_P + \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x} - 1\right) \tag{136}$$

# Inflation in continuous time

Define date $t + dt$ inflation as the net percentage change in the price level between dates $t$ and $t + dt$ per unit time, i.e.

$$\pi_{t+dt} = \frac{P_{t+dt} - P_t}{P_t} \frac{1}{dt} \tag{137}$$

$$\pi_t + d\pi_t = \frac{1}{P_t} \frac{dP_t}{dt} \tag{138}$$

Taking the continuous time limit

$$\pi_t = \frac{\dot{P}_t}{P_t} \; (= \mu_{P,t}) \tag{139}$$

Relationship between inflation and output gap

$$\rho\pi = \dot{\pi} + \frac{\epsilon - 1}{\theta} \left( e^{(1+\varphi)x} - 1 \right) \tag{140}$$

# Insights from New Keynesian Phillip's Curve

$$\rho\pi = \dot{\pi} + \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x} - 1\right) \tag{141}$$

- Because household dislikes work there is a tradeoff between more output and utility, so there is an optimal output path.
- It is optimal to keep inflation and the output gap as small as possible
- Traditional Phillip's Curve linkes inflation to employment – NKPC also tells about this link

$$\rho\pi = \dot{\pi} + \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)(n-n^n)} - 1\right) \tag{142}$$

# Nominal Interest Rate Rules I

- Make nominal interest rate rule depend on current inflation and output gap

$$i_t = v(\pi_t, x_t) \tag{143}$$

- Obtain a system of ordinary differential equations to pin down outgap and inflation

$$r^n + \dot{x} + \pi = v(\pi, x) \tag{144}$$

$$\rho\pi = \dot{\pi} + \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x} - 1\right), \tag{145}$$

  where $r^n = \rho + \dot{a}$ is the natural interest rate.

- Linearization takes us to 3 equation model: $e^{(1+\varphi)x} - 1 \approx (1 + \varphi)x$

# Nominal Interest Rate Rules II

- Suppose that $v(\pi, x) = a + \phi_\pi \pi + \phi_x x$. Then

$$r^n + \dot{x} = (\phi_\pi - 1)\pi + \phi_x x \tag{146}$$

$$\rho\pi = \dot{\pi} + \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x} - 1\right), \tag{147}$$

  where $r^n = \rho + \dot{a}$ is the natural interest rate.

- Transversality condition

$$\theta \lim_{t\to\infty} e^{-\rho t}\pi_t P_t Y_t = \theta P_0 \lim_{t\to\infty} e^{-\rho t}\pi_t e^{\int_0^t \pi_u du} e^{y_t^n + x_t} \tag{148}$$

# Summary

- Central Bank's Balance Sheet and Monetary Policy
- Some Deterministic Control Theory!
- Derived Basic New Keynesian Model
- Still need to apply Basic New Keynesian Model!