Lecture 2: Central Banks, Monetary Policy and Risk

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The Central Bank's Balance Sheet

- Central Banks do not just trade in short-term Government bond markets.
- They also trade in longer term Government debt in an attempt to influence the long end of the yield curve.
- Central banks also hold risky assets, such as mortgage-backed securities and in some cases even equity, which impacts risk premia
- The money used to purchase securities can be created electronically by a Central Bank

Quantitative Easing – when a central bank creates money electronically and uses it to purchase securities from financial institutions

Central Bank's Budget Constraint

$$H_t^{CB} = W_{B,t}^{CB} + \sum_{i=1}^{N} W_{i,t}^{CB} - \frac{M_t}{P_t} \tag{1}$$

$$H_{t+dt}^{CB} = W_{B,t}^{CB}(1 + r_t dt) + \sum_{i=1}^{N} W_{i,t}^{CB}(1 + dR_{i,t}) - \frac{M_t}{P_t}(1 - \pi_t dt) + (T_t - G_t)dt \tag{2}$$

$$dH_t^{CB} = W_{B,t}^{CB} r_t dt + \sum_{i=1}^{N} W_{i,t}^{CB} dR_{i,t} + \frac{M_t}{P_t}\pi_t dt + (T_t - G_t)dt \tag{3}$$

$$dH_t^{CB} = H_t^{CB} r_t dt + \sum_{i=1}^{N} W_{i,t}^{CB}(dR_{i,t} - r_t dt) + \frac{M_t}{P_t}(r_t + \pi_t)dt + (T_t - G_t)dt \tag{4}$$

Typical Stochastic Optimal Control Problem I

- $t \in \mathcal{T} = [0, \infty)$
- We have a 1-d state[1], s, which evolves over time according to the following stochastic law of motion

$$ds(t) = f(s(t), c(t))dt + \sigma(s(t), c(t))dZ(t) \tag{5}$$

- $Z(t)$ is a standard Brownian motion under physical probability measure $\mathbb{P}$
- The starting value of the state is given by $s(0) = s_0$. The future values of the state will depend on the control variable u, which is also 1-d.
- An agent chooses the path of the control, $c(t)_{t \in \mathcal{T}}$. Her objective is to maximize the discounted value of some flow function. At time-t, the flow function is given by

$$u(s(t), c(t)) \tag{6}$$

- With a constant discount rate ρ, the agent's objective is given by

$$J(s_0) = \sup_{c(t)_{t \in \mathcal{T}}} E_0 \int_0^{\infty} e^{-\rho t} u(s(t), c(t))dt \tag{7}$$

Typical Stochastic Optimal Control Problem II

- Date-t objective function

$$J(t) = J(s(t)) = \sup_{c(u)_{u \geq t}} E_t \int_t^{\infty} e^{-\rho(u-t)} u(s(u), c(u)) du \tag{8}$$

- The maximized objective function is called the **value function**
- What path should the agent choose?

[1] later on we shall deal with multidimensional states

Hamilton-Jacobi-Bellman Equation & Stochastic Maximum Principle

- HJB equation

$$0 = \sup_{c(t)} u(s(t), c(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c(t)) + \frac{1}{2} J''(s(t))(\sigma(s(t), c(t)))^2 \quad (9)$$

- Stochastic Maximum Principle

$$\mathcal{H}(s(t), c(t), \lambda(t), \phi(t)) = u(s(t), c(t)) + \lambda(t) f(s(t), c(t)) + \frac{1}{2}\phi(t)\sigma(s(t), c(t))^2 \quad (10)$$

$$\mathcal{H}_c(s(t), c(t), \lambda(t), \phi(t)) = 0, \text{ given } s_0 \quad (11)$$

$$\mathcal{H}_s(s(t), c(t), \lambda(t), \phi(t))dt + d\lambda(t) - \rho\lambda(t)dt = \phi(s(t))\sigma(s(t), c(t))dZ(t) \quad (12)$$

$$E_t[e^{-\rho(T-t)}\lambda_T] = 0, \quad (13)$$

where $\phi(s(t)) = \frac{\partial\lambda(t)}{\partial s(t)}$

Nonlinear differential equation from HJB Equation

The value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{c(t)} u(s(t), c(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c(t)) + \frac{1}{2} J''(s(t))(\sigma(s(t), c(t)))^2 \tag{14}$$

- The FOC condition of the HJB is

$$u_c(s(t), c(t)) = J'(s(t)) f_c(s(t), c(t)) + J''(s(t))\sigma(s(t), c(t))\sigma_s(s(t), c(t)) \tag{15}$$

- Solving the above equation gives the date-t value of the optimal control in terms of the date-t state and the derivative of the value function.
- To find the value function, substitute the optimal control, $c^*(t)$ into the HJB to get a second order nonlinear ordinary differential equation (no longer need the sup)

$$0 = u(s(t), c^*(t)) - \rho J(s(t)) + J'(s(t)) f(s(t), c^*(t)) + \frac{1}{2} J''(s(t))(\sigma(s(t), c^*(t)))^2 \tag{16}$$

- How do we solve the above nonlinear ode?
  - Does not generally have a closed-form solution – need numerical methods
  - Do not have boundary conditions – need a new concept of what a solution to a differential equation is – viscosity solution – shall understand this informally when we study finite difference methods

Bellman's Principle of Optimality

Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)

Key conceptual difference relative to Pontryagin's Maximum Principle

- Think of the control a function of the state.

Heuristic derivation of HJB I

Split the integral for the value function between the next small time interval, $[t, t+dt)$ and the rest of time $[t+dt, \infty)$

We use the infinitesimal dt instead of the small increment Δt, so we can use Ito's Lemma and our usual results from Stochastic Calculus

$$J(s(t)) = \sup_{c(u)_{u \geq t}} E_t \int_t^{t+dt} e^{-\rho(u-t)} u(s(u), c(u)) du \tag{17}$$

$$+ e^{-\rho dt} E_t \left(E_{t+dt} \left[\int_{t+dt}^{\infty} e^{-\rho(u-[t+dt])} u(s(u), c(u)) du \right] \right) \tag{18}$$

Apply Bellman's Principle of Optimality

- Suppose we choose an optimal path at date-t: $(c_u^t(s))_{u \geq t}$
- Suppose we choose an optimal path at date-$\tau > t$: $(c_u^\tau(s))_{u \geq \tau}$
- Principle of Optimality $\Rightarrow (c_u^t(s))_{u \geq t} = (c_u^t(s))_{t \leq u < \tau} \cup (c_u^\tau)_{u \geq \tau}$

If we choose an optimal path for the control at some future date, when considered as functions of the state, the future optimal path is contained within today's – time consistency

For our infinite horizon problem, where the law of motion for the state does not depend explicitly on time, we can go even further: thinking of the control as a map from the state to the reals, it so happens that the map is time invariant, i.e. $\forall t \in \mathcal{T}$, $(c_u^t(s))_{u \geq t}$, for each $u \geq t$, we have $c_u^t(s) = c(s)$

Heuristic derivation of HJB II

Exploiting the Principle of Optimality reveals a recursive structure for the value function

$$J(s(t)) = \sup_{c(u)_{u\in[t,t+dt)}} E_t \Bigg[E_t \int_t^{t+dt} e^{-\rho(u-t)} u(s(u),c(u))du + e^{-\rho dt} \underbrace{\sup_{c(u)_{u\geq t+dt}} E_{t+dt} \int_{t+dt}^{\infty} e^{-\rho(u-[t+dt])} u(s(u),c(u))du}_{J(s(t+dt))} \Bigg] \tag{19}$$

To derive the HJB equation we just need to indulge in some Stochastic Calculus. First observe that

$$E_t \int_t^{t+dt} e^{-\rho(u-t)} u(s(u),c(u))du = u(s(t),c(t))dt + o(dt) \tag{20}$$

$\left[h(t) = o(j(t)) \iff \lim_{t\to 0} \frac{h(t)}{j(t)} = 0\right]$

The E_t is important – it kills off $dZ(t)$ terms!

Heuristic derivation of HJB III

Hence

$$J(s(t)) = \sup_{c(u)_{u\in[t,t+dt)}} u(s(t), c(t))dt + e^{-\rho dt} E_t J(s(t+dt)) + o(dt) \tag{21}$$

Now

$$e^{-\rho dt} = 1 - \rho dt + o(dt), \tag{22}$$

and so

$$J(s(t)) = \sup_{c(u)_{u\in[t,t+dt)}} u(s(t), c(t))dt + E_t J(s(t+dt)) - \rho dt E_t J(s(t+dt)) + o(dt) \tag{23}$$

Furthermore $dt E_t J(s(t+dt)) = dt J(s(t)) + o(dt)$, and so

$$J(s(t)) = \sup_{c(u)_{u\in[t,t+dt)}} u(s(t), c(t))dt + E_t J(s(t+dt)) - \rho dt J(s(t)) + o(dt) \tag{24}$$

Heuristic derivation of HJB IV

Now

$$J(s(t+dt)) = J(s(t)) + dJ(s(t)) \tag{25}$$

$$= J(s(t)) + J'(s(t))E_t ds(t) + \frac{1}{2}J''(s(t))E_t[(ds(t))^2] + o(dt) \tag{26}$$

$$= J(s(t)) + J'(s(t))f(s(t),c(t))dt + \frac{1}{2}J''(s(t))(\sigma(s(t),c(t)))^2 + o(dt) \tag{27}$$

$$\tag{28}$$

and so

$$0 = \sup_{c(t)} \left(u(s(t),c(t)) - \rho J(s(t)) + J'(s(t))f(s(t),c(t)) + \frac{1}{2}J''(s(t))(\sigma(s(t),c(t)))^2 \right) dt + o(dt) \tag{29}$$

In the continuous time limit, we obtain

$$0 = \sup_{c(t)} u(s(t),c(t)) - \rho J(s(t)) + J'(s(t))f(s(t),c(t)) + \frac{1}{2}J''(s(t))(\sigma(s(t),c(t)))^2 \tag{30}$$

Stochastic Maximum Principle and Forward-Backward Sde's I

For the deterministic case, it was fairly straightforward to see the connection between Pontrayagin's Maximum Principle and the Hamilton-Jacobi-Bellman Equation. In the stochastic case, the mathematics is more challenging, so I give a heuristic overview.

The main idea of interest to us is that the Hamiltonian needs to be adjusted for risk – this is very intuitive for financial economists. In the deterministic case, the Hamiltonian included the 'expected return' given a particular control. In the stochastic case, we need to subtract the 'variance'.

The Hamiltonian now takes the form

$$\mathcal{H}(s(t), c(t), \lambda(t), \phi(t)) = u(s(t), c(t)) + \lambda(t) f(s(t), c(t)) + \frac{1}{2}\phi(t)\sigma(s(t), c(t))^2 \qquad (31)$$

As before I make the identification, $\lambda(t) = J'(s(t))$. I make the additional identification $\phi(t) = J''(s(t))$. For problems with a well defined maximum, $J''(s(t)) < 0$, so the term $\frac{1}{2}\phi(t)\sigma(s(t), c(t))^2$ is a downward risk adjustment, in accordance with our portfolio theory intuition.

Stochastic Maximum Principle and Forward-Backward Sde's II

$$\mathcal{H}_c(s(t), c(t), \lambda(t), \phi(t)) = 0, \text{ given } s_0 \tag{32}$$

$$\mathcal{H}_s(s(t), c(t), \lambda(t), \phi(t))dt + d\lambda(t) - \rho\lambda(t)dt = \frac{\partial\lambda(t)}{\partial s(t)}\sigma(s(t), c(t))dZ(t),\ E_t[e^{-\rho(T-t)}\lambda_T] = 0, \tag{33}$$

Equation (32) is a forward sde, while (33) is a backward sde. If they cannot be uncoupled they constitute a forward-backward sde (FBSDE) system.

Solving the FBSDE system

- One way to try and get a solution is to take the conditional expectation of the backward sde, treat λ as a function of the state and the use Ito's Lemma to derive a differential equation for λ.
- The differential equation for λ can be used in conjunction with the Feynman-Kac Theorem to obtain a probabilistic solution for λ
- See Ma & Yong (2008) for a thorough treatment of FBSDE's.

Connection between Stochastic Maximum Principle and the Hamilton-Jacobi-Bellman Equation

- Write HJB equation in terms of Hamiltonian
- Identify $\lambda(t) = J'(s(t))$ (useful to observe that $\lambda(t) = \lambda(s(t))$)
- Identify $\phi(t) = J''(s(t))$ [useful to observe that $\phi(t) = \partial\lambda(s(t))/\partial s(t)$]

$$0 = \sup_{c(t)} \mathcal{H}(s(t), c(t), J'(s(t)), J''(s(t))) - \rho J(s(t)) = 0 \tag{34}$$

- FOC of HJB gives us one part of Maximum Principle

$$\mathcal{H}_c(s(t), c(t), J'(s(t))) = 0 \tag{35}$$

- What about $\mathcal{H}_s(s(t), c(t), \lambda(t), \phi(t))dt - d\lambda(t) - \rho\lambda(t)dt + \frac{\partial\lambda(t)}{\partial s(t)}\sigma(s(t), c(t))dZ(t) = 0$?
  - We can also derive this from the HJB!

Start by noting that at the optimum, where $c(t) = c^*(s(t))$, the HJB becomes the following ode

$$0 = \mathcal{H}(s(t), c^*(s(t)), J'(s(t)), J''(s(t))) - \rho J(s(t)) \tag{36}$$

Differentiate the ode wrt to $s(t)$ and use the fact that $\mathcal{H}_c(s(t), c^*(s(t)), J'(s(t)), J''(s(t))) = 0$

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), J'(s(t)), J''(s(t))) + \overbrace{\mathcal{H}_c(s(t), c^*(s(t)), \lambda(t), \phi(t))}^{=0} \frac{\partial c^*(s(t))}{\partial s(t)} \tag{37}$$

$$+ \underbrace{\mathcal{H}_\lambda(s(t), c^*(s(t)), J'(s(t)), J''(s(t)))}_{=f(s(t),c^*(t))} J''(s(t)) \tag{38}$$

$$+ \underbrace{\mathcal{H}_\phi(s(t), c^*(s(t)), J'(s(t)), J''(s(t)))}_{\frac{1}{2}\sigma(s(t),c^*(t))^2} J'''(s(t)) - \rho J'(s(t)) \tag{39}$$

Remember the identification $\lambda(t) = J'(s(t))$, which implies $\frac{\partial \lambda(t)}{\partial s(t)} = J''(s(t))$

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), \lambda(t)) + f(s(t), c^*(t)) \frac{\partial \lambda(t)}{\partial s(t)} + \frac{1}{2}\sigma(s(t), c^*(t))^2 \frac{\partial^2 \lambda(t)}{\partial s(t)^2} - \rho \lambda(t) \tag{40}$$

Noting that

$$d\lambda(t) = \frac{\partial \lambda(t)}{\partial s(t)} ds(t) + \frac{1}{2}\frac{\partial^2 \lambda(t)}{\partial s(t)^2}(ds(t))^2 \tag{41}$$

$$= \left[\frac{\partial \lambda(t)}{\partial s(t)} f(s(t), c^*(t)) + \frac{1}{2}\frac{\partial^2 \lambda(t)}{\partial s(t)^2}\sigma(s(t), c^*(t))^2\right] dt \tag{42}$$

$$+ \frac{\partial \lambda(t)}{\partial s(t)}\sigma(s(t), c^*(t))dZ(t), \tag{43}$$

we obtain

$$0 = \mathcal{H}_s(s(t), c^*(s(t)), \lambda(t)) + E_t \frac{d\lambda(t)}{dt} - \rho\lambda(t) \tag{44}$$

and hence by treating $\lambda(t)$ as a function of $s(t)$ and using Ito's Lemma

$$\frac{\partial \lambda(t)}{\partial s(t)}\sigma(s(t), c^*(t))dZ(t) = \mathcal{H}_s(s(t), c^*(s(t)), \lambda(t))dt + d\lambda(t) - \rho\lambda(t)dt \tag{45}$$

Model

Look at a simple version of the Cox-Ingersoll-Ross production economy (see Cox, Ingersoll & Ross (1985))

- Stochastic optimal control problem

$$\sup_{(C_u)_{u\geq t}} E_t \int_t^{\infty} e^{-\rho(u-t)} \frac{C_u^{1-\gamma}}{1-\gamma} du \tag{46}$$

s.t.

$$dK_t = (AK_t - C_t)dt + \sigma K_t dZ_t. \tag{47}$$

Underlying economics I

- Household – standard power utility
- Capital accumulation equation
  - General capital accumulation equation

$$K_{t+dt} = (K_t + I_t dt)e^{-\delta dt} + \sigma_t K_t dZ_t \tag{48}$$

    - Current capital stock is augmented by an investment flow, giving $K_t + I_t dt$
    - There is depreciation at the rate δ
    - The is a zero-mean additive shock to the capital stock
  - In the continuous time limit, we obtain

$$dK_t = (I_t - \delta K_t)dt + \sigma_t K_t dZ_t \tag{49}$$

  - Impose market clearing

$$Y_t = C_t + I_t \tag{50}$$

    - The rate of output is sum of the consumption rate and the investment flow

Underlying economics II

- Obtain capital accumulation equation in terms of capital stock level, consumption rate and output flow

$$dK_t = (Y_t - \delta K_t - C_t)dt + \sigma_t K_t dZ_t \tag{51}$$

- Assume a linear output function, $Y_t = \alpha K_t$

$$dK_t = (AK_t - C_t)dt + \sigma_t K_t dZ_t,\ A = \alpha - \delta \tag{52}$$

- Assume constant σ_t, i.e. $\sigma_t = \sigma$

$$dK_t = (AK_t - C_t)dt + \sigma K_t dZ_t \tag{53}$$

Use log variables I

- Stochastic optimal control problem

$$\sup_{(C_u)_{u\geq t}} E_t \int_t^\infty e^{-\rho(u-t)} \frac{C_u^{1-\gamma}}{1-\gamma} du \tag{54}$$

s.t.

$$dK_t = (AK_t - C_t)dt + \sigma K_t dZ_t. \tag{55}$$

- Using log variables

$$\sup_{(c_u)_{u\geq t}} E_t \int_t^\infty e^{-\rho(u-t)} \frac{e^{(1-\gamma)c_u}}{1-\gamma} du \tag{56}$$

$$dk_t = (\mu - e^{c_t - k_t})dt + \sigma dZ_t \tag{57}$$

where $\mu = A - \frac{1}{2}\sigma^2$

HJB Equation and Maximum Principle I

- HJB equation

$$0 = \sup_{c_t} \mathcal{H}(k_t, c_t, \partial_k J_t, \partial_{kk} J_t) - \rho J_t \tag{58}$$

where

$$\mathcal{H}(k_t, c_t, \Lambda_t, \Phi_t) = \frac{e^{(1-\gamma)c_t}}{1-\gamma} + (\mu - e^{c_t - k_t})\Lambda_t + \frac{1}{2}\sigma^2 \Phi_t \tag{59}$$

- FOC for c

$$\mathcal{H}_c = 0 \tag{60}$$

$$e^{(1-\gamma)c} = e^{c-k}\Lambda \tag{61}$$

$$e^{-\gamma c} = e^{-k}\Lambda \tag{62}$$

$$\Lambda = e^{k-\gamma c} \tag{63}$$

HJB Equation and Maximum Principle II

$$e^c = e^{\frac{1}{\gamma}k}\Lambda^{-\frac{1}{\gamma}} \tag{64}$$

- Differentiate HJB eqn wrt k

$$0 = \mathcal{H}(k_t, c_t^*, \partial_k J_t, \partial_{kk} J_t) - \rho J_t \tag{65}$$

$$0 = \mathcal{H}_c \frac{\partial c^*}{\partial k} + \mathcal{H}_k + \mathcal{H}_\Lambda \frac{\partial \Lambda}{\partial k} + \mathcal{H}_\Phi \frac{\partial \Phi}{\partial k} - \rho\Lambda \tag{66}$$

$$0 = \mathcal{H}_k + \mathcal{H}_\Lambda \frac{\partial \Lambda}{\partial k} + \mathcal{H}_\Phi \frac{\partial \Phi}{\partial k} - \rho\Lambda \tag{67}$$

$$0 = \mathcal{H}_k + E_t \frac{d\Lambda}{dt} - \rho\Lambda \tag{68}$$

$$d\Lambda = (\rho\Lambda - \mathcal{H}_k)dt + \Phi\sigma dZ \tag{69}$$

$$\tag{70}$$

HJB Equation and Maximum Principle III

where $\Phi = \frac{\partial \Lambda}{\partial k}$. Last step – used Ito's Lemma

$$d\Lambda = \left(\rho - e^{-\left(1-\frac{1}{\gamma}\right)k}\Lambda^{-\frac{1}{\gamma}}\right)\Lambda dt + \Phi\sigma dZ \tag{71}$$

Deriving FBSDE I

- forward-backward sde (FBSDE) system
  - forward sde

$$dk_t = \left(\mu - e^{-\left(1-\frac{1}{\gamma}\right)k_t}\Lambda_t^{-\frac{1}{\gamma}}\right)dt + \sigma dZ_t, \text{ given } k_0 \tag{72}$$

  - backward sde

$$d\Lambda_t = \left(\rho - e^{-\left(1-\frac{1}{\gamma}\right)k_t}\Lambda_t^{-\frac{1}{\gamma}}\right)\Lambda_t dt + \Phi_t \sigma dZ, \; \Phi_t = \frac{\partial \Lambda_t}{\partial k_t}, \; \lim_{T\to\infty} e^{-\rho T} E_0[\Lambda_T] = 0 \tag{73}$$

Deriving FBSDE II

- Change of variables: $\lambda = \ln \Lambda$
  - forward sde

$$dk_t = \left(\mu - e^{-\left(1-\frac{1}{\gamma}\right)k_t}\Lambda_t^{-\frac{1}{\gamma}}\right)dt + \sigma dZ_t, \text{ given } k_0 \tag{74}$$

  - backward sde

$$d\lambda_t = \left(\rho - \frac{1}{2}\left(\frac{\partial \lambda_t}{\partial k_t}\right)^2 \sigma^2 - e^{-\left(1-\frac{1}{\gamma}\right)k_t} e^{-\frac{1}{\gamma}\lambda_t}\right)dt + \frac{\partial \lambda_t}{\partial k_t}\sigma dZ \tag{75}$$

$$\lim_{T\to\infty} e^{-\rho T} E_0[e^{\lambda_T}] = 0 \tag{76}$$

Solving FBSDE system I

- FBSDE
  - forward sde

$$dk_t = \left(\mu - e^{-\left(1-\frac{1}{\gamma}\right)k_t} e^{-\frac{1}{\gamma}\lambda_t}\right) dt + \sigma dZ_t, \text{ given } k_0 \tag{77}$$

  - backward sde

$$d\lambda_t = \left(\rho - \frac{1}{2}\left(\frac{\partial \lambda_t}{\partial k_t}\right)^2 \sigma^2 - e^{-\left(1-\frac{1}{\gamma}\right)k_t} e^{-\frac{1}{\gamma}\lambda_t}\right) dt + \frac{\partial \lambda_t}{\partial k_t}\sigma dZ, \quad \lim_{T\to\infty} e^{-\rho T} E_0[e^{\lambda_T}] \tag{78}$$

- Ansatz: $\lambda_t = a_0 + a_1 k_t$

$$dk_t = \left(\mu - e^{-\left(1-\frac{1}{\gamma}\right)k_t} e^{-\frac{1}{\gamma}(a_0+a_1 k_t)}\right) dt + \sigma dZ_t \tag{79}$$

$$d\lambda_t = \left(\rho - \frac{1}{2}a_1^2\sigma^2 - e^{-\left(1-\frac{1}{\gamma}\right)k_t} e^{-\frac{1}{\gamma}(a_0+a_1 k_t)}\right) dt + a_1\sigma dZ \tag{80}$$

Solving FBSDE system II

- Since $d\lambda = a_1 dk$, we have

$$\rho - \frac{1}{2}a_1^2\sigma^2 - a_1\mu = (1 - a_1)e^{-\left(1-\frac{1}{\gamma}\right)k-\frac{1}{\gamma}a_1 k}e^{-\frac{1}{\gamma}a_0} \tag{81}$$

- Two possibilities
  - RHS is function of $k \Rightarrow$ no solution for most parameter values
  - RHS is constant $\Rightarrow a_1 = 1 - \gamma$ &

$$e^{-\frac{1}{\gamma}a_0} = \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\left(A - \frac{1}{2}\gamma\sigma^2\right) \tag{82}$$

Solving FBSDE system III

- Solution

$$\lambda_t = -\gamma \ln \left[\frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\left(A - \frac{1}{2}\gamma\sigma^2\right) \right] + (1-\gamma)k_t \tag{83}$$

$$\Lambda_t = \left[\frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\left(A - \frac{1}{2}\gamma\sigma^2\right) \right]^{-\gamma} K_t^{1-\gamma} \tag{84}$$

and

$$\frac{dK_t}{K_t} = \left[\frac{1}{\gamma}(A-\rho) + \left(1 - \frac{1}{\gamma}\right)\frac{1}{2}\gamma\sigma^2 \right] dt + \sigma dZ_t, \text{ given } K_0 \tag{85}$$

$$K_t = K_0 e^{\left[\frac{1}{\gamma}(A-\rho) + \left(1-\frac{1}{\gamma}\right)\frac{1}{2}\gamma\sigma^2\right]t} e^{-\frac{1}{2}\sigma^2 t + \sigma Z_t} \tag{86}$$

Optimal consumption rate

$$C_t = \left[\frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\left(A - \frac{1}{2}\gamma\sigma^2\right) \right] K_t \tag{87}$$

Stochastic Growth Model

$$J_t = \sup_{(C_u)_{u \geq t}} E_t \left[\int_t^{\infty} f(C_u, J_u) du \right] \tag{88}$$

where

$$f(c, v) = \rho (h^{-1}(v))^{1-\gamma} u \left(\frac{c}{h^{-1}(v)} \right) \tag{89}$$

$$h(x) = \frac{x^{1-\gamma}}{1-\gamma} \tag{90}$$

$$u(x) = \frac{x^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}} \tag{91}$$

s.t.

$$dK_t = (I_t - \delta K_t) dt + \sigma K_t dZ_t, \tag{92}$$

where

$$I_t + C_t = Y_t \tag{93}$$

and

$$Y_t = A K_t^{\alpha} \tag{94}$$

HJB Equation

HJB equation

$$\sup_{C_t} f(C_t, J_t) + J'(K_t)(AK_t^{\alpha} - \delta K_t - C_t) + \frac{1}{2}\sigma^2 K_t^2 J''(K_t) \tag{95}$$

FOC (drop excessive notation, such as t-subscript and K-argument)

$$\rho(h^{-1}(J))^{-\left(\gamma-\frac{1}{\psi}\right)} C^{-\frac{1}{\psi}} = J' \tag{96}$$

Make a transformation

$$J = \frac{e^{(1-\gamma)g(k)}}{1-\gamma} \tag{97}$$

$$k = \ln K \tag{98}$$

Note that while $J' = \partial_K J$, $g' = \partial_k g$, so we obtain

$$J' = e^{(1-\gamma)g-k} g' \tag{99}$$

$$J'' = e^{(1-\gamma)g-2k}(g'' - (\gamma-1)(g')^2 - g') \tag{100}$$

$$f(C, J) = \rho e^{(1-\gamma)g} u\left(\frac{C}{e^g}\right) \tag{101}$$

The HJB reduces to the following nonlinear second order ode

$$0 = \rho u\left(C^* e^{-g}\right) + \left(Ae^{-(1-\alpha)k} - \delta - C^* e^{-k}\right) g'(k) + \frac{1}{2}\sigma^2(g'' - (\gamma-1)(g')^2 - g'), \tag{102}$$

where C^* denotes the optimal consumption rate. From the FOC, we can see that

$$C^* = [\rho e^k (g')^{-1}]^{\psi} e^{(1-\psi)g} \tag{103}$$

We want to isolate the nonlinear terms
From the FOC

$$C^* e^{-g} = [\rho e^k e^{-g} (g')^{-1}]^{\psi}, \tag{104}$$

Hence

$$(C^* e^{-g})^{1-\frac{1}{\psi}} = [\rho e^k e^{-g} (g')^{-1}]^{\psi - 1} \tag{105}$$

and so

$$\rho u(C^* e^{-g}) - C^* e^{-k} g = \frac{\rho^{\psi} [e^k e^{-g} (g')^{-1}]^{\psi-1} - \rho}{1 - \frac{1}{\psi}} - \rho^{\psi} [e^k e^{-g} (g')^{-1}]^{\psi - 1} \tag{106}$$

$$= \frac{\rho^{\psi} [e^k e^{-g} (g')^{-1}]^{\psi-1}}{\psi - 1} - \frac{\rho\psi}{\psi - 1} \tag{107}$$

$$= \frac{C^* e^{-g} g'}{\psi - 1} - \frac{\rho\psi}{\psi - 1} \tag{108}$$

The ode can thus be rewritten as

$$0 = \left(\beta\psi - \left\{(\psi - 1)\left[A e^{-(1-\alpha)k} - \left(\delta + \frac{1}{2}\sigma^2\right)\right] + C^* e^{-g}\right\}\partial_k - \frac{1}{2}(\psi - 1)\sigma^2 \partial_{kk}\right) g \tag{109}$$

$$+ \frac{1}{2}\sigma^2 (\gamma - 1)(\psi - 1)(g')^2 \tag{110}$$

Approximate Loglinear Solution I

We now linearize the ode around the deterministic steady-state (dss).

$$\hat{k} = k - k|_{dss} \tag{111}$$

Observe that

$$e^{-(1-\alpha)k} = e^{-(1-\alpha)(k_{dss}+\hat{k})} = e^{-(1-\alpha)k_{dss}}[1-(1-\alpha)\hat{k}] + O(\hat{k}^2) \tag{112}$$

and

$$C^* e^{-g} = e^{c^*-g} = e^{c_{dss}-g_{dss}} + [c^* - g - (c_{dss} - g_{dss})]e^{c_{dss}-g_{dss}} + O([c^* - g - (c_{dss} - g_{dss})]^2) \tag{113}$$

$$= [1 - (c_{dss} - g_{dss})]e^{c_{dss}-g_{dss}} + (c^* - g)e^{c_{dss}-g_{dss}} + O([c^* - g - (c_{dss} - g_{dss})]^2) \tag{114}$$

From

$$C^* e^{-g} = [\rho e^k e^{-g}(g')^{-1}]^{\psi}, \tag{115}$$

we have

$$c^* - g = \psi[\ln\rho + k - g - \ln\partial_{\hat{k}} g] \tag{116}$$

and so

$$C^* e^{-g} = [1 - (c_{dss} - g_{dss})]e^{c_{dss}-g_{dss}} + \psi[\ln\rho + k - g - \ln\partial_{\hat{k}} g]e^{c_{dss}-g_{dss}} \tag{117}$$

$$+ O([c^* - g - (c_{dss} - g_{dss})]^2) \tag{118}$$

Approximate Loglinear Solution II

Now suppose that

$$g = g_{dss} + a\,\hat{k} + O(\hat{k}^2) \tag{119}$$

I can therefore reduce the nonlinear ode for g into an equation which is linear in $\hat{k}$. By comparing coefficients, I can obtain two nonlinear simultaneous equations in g_{dss} and a. This is best done in mathematica!

$$0 = \left(\beta\psi - \left\{(\psi - 1)\left[Ae^{-(1-\alpha)k_{dss}}(1-(1-\alpha)\hat{k}) - \left(\delta + \frac{1}{2}\sigma^2\right)\right] + C^* e^{-g}\right\}\partial_k \tag{120}$$

$$-\frac{1}{2}(\psi - 1)\sigma^2\partial_{kk}\right) g \tag{121}$$

$$+\frac{1}{2}\sigma^2(\gamma - 1)(\psi - 1)(g')^2 \tag{122}$$

Numerical Solution I

For the numerical solution, I shall solve for J, but use $k = \ln K$ as the underlying state variable. (this turns out to be better than (g, k) and (J, K))

$$\partial_K J = e^{-k} \partial_k J \tag{123}$$

$$\partial_{KK} J = -e^{-2k} \partial_k J + e^{-2k} \partial_{kk} J \tag{124}$$

The HJB equation thus becomes

$$\sup_C f(C, J) + \left[A e^{-(1-\alpha)k} - \left(\delta + \frac{1}{2}\sigma^2 \right) - C e^{-k} \right] \partial_k J + \frac{1}{2}\sigma^2 \partial_{kk} J \tag{125}$$

The FOC is

$$\rho (h^{-1}(J))^{-\left(\gamma - \frac{1}{\psi}\right)} C^{-\frac{1}{\psi}} = e^{-k} \partial_k J \tag{126}$$

Hence

$$\rho^{\psi} (h^{-1}(J))^{-\psi\left(\gamma - \frac{1}{\psi}\right)} C^{-1} = e^{-\psi k} (\partial_k J)^{\psi} \tag{127}$$

$$C^{-1} = \rho^{-\psi} e^{-\psi k} (\partial_k J)^{\psi} (h^{-1}(J))^{\psi\left(\gamma - \frac{1}{\psi}\right)} \tag{128}$$

Numerical Solution II

$$f(C,J) = \frac{\rho[h^{-1}(J)]^{1-\gamma}[C^{1-\frac{1}{\psi}}h^{-1}(J)]^{\frac{1}{\psi}-1} - 1]}{1-\frac{1}{\psi}} \tag{129}$$

$$= \frac{\rho[h^{-1}(J)]^{-(\gamma-\frac{1}{\psi})}C^{1-\frac{1}{\psi}} - \rho[h^{-1}(J)]^{1-\gamma}}{1-\frac{1}{\psi}} \tag{130}$$

$$= \frac{Ce^{-k}\partial_k J - \rho(1-\gamma)J}{1-\frac{1}{\psi}} \tag{131}$$

- Want to solve numerically

$$\frac{Ce^{-k}\partial_k J - \rho(1-\gamma)J}{1-\frac{1}{\psi}} + \left[Ae^{-(1-\alpha)k} - \left(\delta + \frac{1}{2}\sigma^2\right) - Ce^{-k}\right]\partial_k J + \frac{1}{2}\sigma^2\partial_{kk}J \tag{132}$$

- We shall use a **finite difference method**.
- Approximate the function J at $N+1$ equispaced points $k_1, \ldots, k_{N+1}$, where $k_{n+1} = k_n + \Delta k$.

$$J(n) = J(k_n) \tag{133}$$

- We are approximating the **continuous time stochastic process** k, where the underlying state space is continuous (it's actually $\mathbb{R}$) via a **continuous time Markov chain**, where the underlying state space is discrete and bounded (it's $\{k_1, \ldots, k_{N+1}\}$).
- We approximate the first derivative in a very special way, called an **upwind approximation** common in fluid dynamics. Below the steady state, when $E_t[dk_t] > 0$, k is increasing with time if there are no shocks, so we use a **forward first difference** to approximate $\partial_k J$

$$\partial_k^F J(k_n) = \frac{J(n+1) - J(n)}{\Delta k} \tag{134}$$

Above the steady state, when $E_t[dk_t] < 0$, k is decreasing with time if there are no shocks, so we use a **backward first difference** to approximate $\partial_k J$

$$\partial_k^B J(k_n) = \frac{J(n) - J(n-1)}{\Delta k} \tag{135}$$

- The optimal control C is a function of the first derivative of the value function, so using the upwind approximation

$$C^U(n) = C^F(n) I_{\{D^F(n)>0\}} + C^{ss}(n) I_{\{D^F(n)<0, D^B(n)>0\}} + C^B(n) I_{\{D^B(n)<0\}} \tag{136}$$

$$C^F(n) = \rho^{\psi}(h^{-1}(J))^{-\theta\psi} e^{\psi k}[\partial_k^F J(n)]^{-\psi} \tag{137}$$

$$C^B(n) = \rho^{\psi}(h^{-1}(J))^{-\theta\psi} e^{\psi k}[\partial_k^B J(n)]^{-\psi}; \tag{138}$$

where

$$D^F(n) = Ae^{-(1-\alpha)k(n)} - \left(\delta + \frac{1}{2}\sigma^2\right) - C^F(n)e^{-k_n} \tag{139}$$

$$D^B(n) = Ae^{-(1-\alpha)k(n)} - \left(\delta + \frac{1}{2}\sigma^2\right) - C^B(n)e^{-k_n} \tag{140}$$

- Here we are making use of the steady-state, which we are essentially using as a boundary condition.
- For the second derivative

$$\partial_{kk} J = \frac{J(n+1) - 2J(n) - J(n-1)}{(\Delta k)^2} \tag{141}$$

- Discretizing as above, we obtain

$$0 = f(n) + D^F(n) \cdot I_{\{D^F(n)>0\}} \frac{J(n+1) - J(n)}{\Delta k} \tag{142}$$

$$+ D^B(n) \cdot I_{\{D^B(n)>0\}} \frac{J(n) - J(n-1)}{\Delta k} \tag{143}$$

$$+ \frac{1}{2}\sigma^2 \frac{J(n+1) - 2J(n) - J(n-1)}{(\Delta k)^2}, \tag{144}$$

where

$$f(n) = f(C^U(n), J(n)) = \frac{C^U(n)e^{-k}\partial_k^U J(n) - \rho(1-\gamma)J(n)}{1 - \frac{1}{\psi}} \tag{145}$$

and

$$C^U(n) = C^F(n)I_{\{D^U(n)>0\}} + C^{ss}(n)I_{\{D^U(n)<0, D^B(n)>0\}} + C^B(n)I_{\{D^B(n)<0\}} \tag{146}$$

$$C^F(n) = \rho^{\psi}(h^{-1}(J))^{-\theta\psi}e^{\psi k}[\partial_k^F J(n)]^{-\psi} \tag{147}$$

$$C^B(n) = \rho^{\psi}(h^{-1}(J))^{-\theta\psi}e^{\psi k}[\partial_k^B J(n)]^{-\psi}; \tag{148}$$

- For clarity, we can write the discretized ode as

$$0 = f(n) + x(n)J(n-1) + y(n)J(n) + z(n)J(n+1) \tag{149}$$

where

$$x(n) = -\frac{D^B(n) \cdot I_{\{D^B(n)<0\}}}{\Delta k} \tag{150}$$

$$y(n) = \frac{D^{B,-}(n) \cdot I_{\{D^B(n)<0\}} - D^{F,+}(n) \cdot I_{\{D^F(n)>0\}}}{\Delta k} \tag{151}$$

$$z(n) = \frac{D^{F,+}(n) \cdot I_{\{D^F(n)>0\}}}{\Delta k}. \tag{152}$$

- We can use an iterative scheme to find J. Suppose we have an approximate solution $\underline{J}^i = (J^i(1), \ldots, J^i(I))^\top$. How can we obtain a new approximation, $\underline{J}^{i+1} = (J^{i+1}(1), \ldots, J^{i+1}(I))^\top$?
- There are two main approaches: an **explicit method** or an **implicit method**
- The explicit method allows to obtain $\underline{J}^{i+1}$ from $\underline{J}^i$ without any matrix inversion

$$\frac{J(n)^{i+1} - J^i(n)}{\Delta} = f(n) + x(n)J(n-1) + y(n)J(n) + z(n)J(n+1) \tag{153}$$

- In matrix form, it is clear that we can obtain $\underline{J}^{i+1}$ directly from $\underline{J}^{i}$ without matrix inversion.

$$\underline{J}^{i+1} = \underline{f}\Delta + (\mathbf{I} + \Delta\mathbf{A})\underline{J}^{i} \tag{154}$$

where $\underline{f} = (f(1), \ldots, f(N+1))^{\top}$ and $\boldsymbol{A}$ is an $N+1$ by $N+1$ matrix with $(y_1, \ldots, y_{N+1})$ along the diagonal $(z_1, \ldots, z_N)$ along the superdiagonal and $(x_2, \ldots, x_{N+1})$ along the subdiagonal and 0's everywhere else.

- We have approximated the continuous time stochastic process for k via a continuous time Markov chain with generator matrix $\mathbf{A}$ (under the physical measure $\mathbb{P}$)
- $\mathbf{P}_k(\Delta) = e^{\Delta\mathbf{A}} \approx \mathbf{I} + \Delta\mathbf{A}$ is the physical transition matrix for the Markov chain used to approximate k. $[\mathbf{P}_k(\Delta)]_{ij}$ is the physical probability that $k_t = k(j)$ at date-$(t+\Delta)$, conditional on $k_t = k(i)$ at date-t
- We could also use an **implicit method**

$$\frac{J^{i+1}(n) - J^{i}(n)}{\Delta} = f(n) + x(n)J^{i+1}(n-1) + y(n)J^{i+1}(n)$$
$$z(n)J^{i+1}(n+1) \tag{155}$$

- In matrix form

$$(\mathbf{I} - \Delta\mathbf{A})\,\underline{J}^{i+1} = \Delta\,\underline{f} + \underline{J}^{i} \tag{156}$$

- We must invert the matrix $\mathbf{I} - \Delta\mathbf{A}$ to obtain tje **updated** solution – this is why the method is **implicit**.
- Which method to use? Implicit methods are more stable in the sense that they converge more rapidly for a given step-size, Δ.

Model I

- Representative household

$$E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du \tag{157}$$

  - C, consumption rate of composite good
  - N, rate of labor supply

- Continuum of firms $i \in [0,1]$ produces differentiated goods

$$Y_t(i) = A_t N_t(i) \tag{158}$$

  - A_t, common exogenous tech level

- Composite good defined by basket

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{1}{1-\frac{1}{\epsilon}}} \tag{159}$$

Model II

- $C_t(i)dt$ is the quantity of good i consumed by the household over the interval $[t, t + dt)$.
- Can show that:
  - nominal expenditure on consumption aggregates nicely

$$P_t C_t = \int_0^1 P_t(i) C_t(i) di \tag{160}$$

where

$$P_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}} \tag{161}$$

$$\forall i \in [0, 1], \ C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} C_t \tag{162}$$

- static intertemporal budget constraint

$$W_0 = E_t \int_0^\infty \Lambda_t \left(C_t - \frac{W_t}{P_t} N_t\right) dt \tag{163}$$

Model III

- Lagrangian

$$\mathcal{L} = E_t \int_0^\infty e^{-\delta t} \left(\ln C_t - \frac{N_t^{1+\varphi}}{1+\varphi} \right) dt - \kappa \int_0^\infty \Lambda_t \left(C_t - \frac{W_t}{P_t} N_t \right) dt \quad (164)$$

- FOC's
  - consumption

$$e^{-\delta t} C_t^{-1} = \kappa \Lambda_t \quad (165)$$

  - labor

$$e^{-\delta t} N_t^{\varphi} = \kappa \Lambda_t \frac{W_t}{P_t} \quad (166)$$

- Implications of FOC's

Model IV

- SDF process

$$e^{-\delta(u-t)}\left(\frac{C_u}{C_t}\right)^{-1}=\frac{\Lambda_u}{\Lambda_t} \tag{167}$$

- consumption-labor

$$N_t^{\varphi}C_t=\frac{W_t}{P_t} \tag{168}$$

Individual Firm's Stochastic Optimal Control Problem I

- Objective of firm j is to set prices in order to maximize firm value net of adjustment costs

$$\sup_{dP_t(j)/dt} E_t \int_t^{\infty} \frac{\Lambda_u^{\$}}{\Lambda_t^{\$}} \left[\Pi_u(j) - \Theta_u(j)\right] du \tag{169}$$

  - real firm value is the above nominal value dividend by P_t, the aggregate price index

- Nominal profit flow function

$$\Pi_t(j) = P_t(j) Y_t(j) - W_t N_t(j) \tag{170}$$

Individual Firm's Stochastic Optimal Control Problem II

- Adjustment cost function

$$\Theta_t = \frac{1}{2}\theta \left(\frac{dP_t(j)/dt}{P_t} \right)^2 P_t Y_t \tag{171}$$

$$\Pi_t(j) = \left[\left(\frac{P_t(j)}{P_t} \right)^{1-\epsilon} - \frac{W_t}{A_t P_t} \left(\frac{P_t(i)}{P_t} \right)^{\epsilon} \right] P_t Y_t \tag{172}$$

- Simpler notation: $x_t = P_t(j)$, $\mu_t = dP_t(j)/dt$
- Stochastic optimal control problem

$$\sup_{\mu_t} E_t \int_t^\infty \frac{\Lambda_u^\$}{\Lambda_t^\$} P_u Y_u \left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u} x_u^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_u}{x_u} \right)^2 \right] du \tag{173}$$

$$\text{s.t. } dx_t = \mu_t dt \tag{174}$$

Individual Firm's Stochastic Optimal Control Problem III

- x is the state variable, which is the price of the good produced by the firm
- μ, the rate of change of the state variable, is the control
- assuming price is locally deterministic, i.e. no volatility

- Rewrite objective function

$$P_t Y_t \sup_{\mu_t} E_t \int_t^\infty \frac{\Lambda_u^\$}{\Lambda_t^\$} \frac{P_u Y_u}{P_t Y_t} \left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u} x_u^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_u}{x_u} \right)^2 \right] du \tag{175}$$

$$P_t Y_t \sup_{\mu_t} E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} \frac{Y_u}{Y_t} \left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u} x_u^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_u}{x_u} \right)^2 \right] du \tag{176}$$

- With log utility over intermediate consumption, $\frac{\Lambda_u Y_u}{\Lambda_t Y_t} = e^{-\delta(u-t)}$

Individual Firm's Stochastic Optimal Control Problem IV

- Real value of firm net of adjustment costs reduces to

$$Y_t \sup_{\mu_t} E_t \int_t^{\infty} e^{-\delta(u-t)} \left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u} x_u^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_u}{x_u} \right)^2 \right] du \quad (177)$$

- Value function

$$J_t = \sup_{\mu_t} E_t \int_t^{\infty} e^{-\int_t^u k_s ds} \left[\left(x_u^{1-\epsilon} - \frac{W_u}{A_u} x_u^{-\epsilon} \right) P_u^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_u}{x_u} \right)^2 \right] du \quad (178)$$

- Hamiltonian

$$\mathcal{H} = \left(x_t^{1-\epsilon} - \frac{W_t}{A_t} x_t^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_t}{x_t} \right)^2 + E_t \left[\frac{dJ_t}{dt} \right] \quad (179)$$

Individual Firm's Stochastic Optimal Control Problem V

- HJB equation

$$\sup_{\mu_t} \mathcal{H} - \delta J_t \tag{180}$$

- FOC wrt μ

$$\mathcal{H}_\mu = 0 \tag{181}$$

$$J_x = \theta \frac{\mu}{x^2} \tag{182}$$

Individual Firm's Stochastic Optimal Control Problem VI

- Derive Stochastic Maximum Principle by diff'ing HJB wrt x at optimum

$$0 = \mathcal{H}_{\mu}\frac{\partial \mu}{\partial x} + \mathcal{H}_x - kJ_x \quad (183)$$

$$0 = \mathcal{H}_x - kJ_x \quad (184)$$

$$0 = \partial_x \left[\left(x_t^{1-\epsilon} - \frac{W_t}{A_t} x_t^{-\epsilon} \right) P_t^{\epsilon-1} - \frac{1}{2}\theta \left(\frac{\mu_t}{x_t} \right)^2 \right] - kJ_x - E_t \left[\frac{d}{dt} J_x \right] \quad (185)$$

- Define $\Phi = J_x$

$$0 = (\epsilon - 1)x^{-\epsilon} \left[\frac{\epsilon}{\epsilon - 1}\frac{W}{Ax} - 1 \right] P^{\epsilon-1} + \theta\mu^2 x^{-3} + E_t \left[\frac{d\Phi}{dt} \right] - \delta\Phi \quad (186)$$

$$\Phi = \theta\frac{\mu}{x^2} \quad (187)$$

Individual Firm's Stochastic Optimal Control Problem VII

- Symmetric equilibrium, where $x = P$. Define $\pi = \mu/P$

$$0 = (\epsilon - 1)\left[\frac{\epsilon}{\epsilon - 1}\frac{W}{AP} - 1\right]P^{-1} + \theta\pi^2 P^{-1} + E_t\left[\frac{d\Phi}{dt}\right] - \delta\Phi \tag{188}$$

$$\Phi = \theta\pi P^{-1} \tag{189}$$

- Symmetric equilibrium, where $x = P$. Define $\pi = \mu/P$
- With no adjustment costs, i.e. $\theta = 0$, then

$$P_t = \frac{\epsilon}{\epsilon - 1}\frac{W_t}{A_t} \tag{190}$$

Individual Firm's Stochastic Optimal Control Problem VIII

- Use the second equation to eliminate Φ in the first equation.

$$d\Phi = \theta d\left(\frac{\pi}{P}\right) \tag{191}$$

$$= \theta P^{-1}\left\{d\pi - \pi\frac{dP}{P} - d\pi\frac{dP}{P} + \pi\left(\frac{dP}{P}\right)^2\right\} \tag{192}$$

- Since P has zero diffusion term, we obtain

$$d\Phi = \theta P^{-1}\left(d\pi - \pi^2 dt\right) \tag{193}$$

- Hence obtain real wage rate in terms of technology level and inflation

$$\frac{\epsilon - 1}{\theta}\left(\frac{\epsilon}{\epsilon - 1}\frac{W_t}{P_t A_t} - 1\right) + E_t\left[\frac{d\pi_t}{dt}\right] - \delta\pi_t = 0, \tag{194}$$

Bond Pricing

- Nominal interest rate

$$i_t = r_t + \pi_t \tag{195}$$

- Real interest rate

$$r_t = \delta + \mu_{Y,t} - \sigma_{Y,t}^2, \tag{196}$$

  - Conditional moments of output

$$E_t\left[\frac{dY_t}{Y_t}\right] = \mu_{Y,t}dt \tag{197}$$

$$E_t\left[\left(\frac{dY_t}{Y_t}\right)^2\right] = \sigma_{Y,t}^2 dt \tag{198}$$

Summary

- From household's FOC's

$$e^{-\delta u}\left(\frac{C_u}{C_t}\right)^{-1} = \frac{\Lambda_u}{\Lambda_t} \tag{199}$$

$$N_t^{\varphi} C_t = \frac{W_t}{P_t} \tag{200}$$

- From firm's FOC's

$$\frac{\epsilon - 1}{\theta}\left(\frac{\epsilon}{\epsilon - 1}\frac{W_t}{P_t A_t} - 1\right) + E_t\left[\frac{d\pi_t}{dt}\right] - \delta\pi_t = 0, \tag{201}$$

- From bond pricing

$$i_t = r_t + \pi_t \tag{202}$$

$$r_t = \delta + \mu_{Y,t} - \sigma^2_{Y,t} \tag{203}$$

Solving for Eqm

- Relating output to inflation

$$\frac{W}{PA} = \frac{N^{\varphi}C}{A} = \frac{N^{\varphi}NA}{A} = N^{1+\varphi} = e^{(1+\varphi)(y-a)} \tag{204}$$

and so

$$\frac{\epsilon - 1}{\theta}\left(\frac{\epsilon}{\epsilon - 1}e^{(1+\varphi)(y_t - a_t)} - 1\right) + E_t\left[\frac{d\pi_t}{dt}\right] - \delta\pi_t = 0, \tag{205}$$

New Keynesian Phillips Curve

- By imposing market clearing, we can show that

$$0 = \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x_t} - 1\right) + E_t\left[\frac{d\pi_t}{dt}\right] - \delta\pi_t, \tag{206}$$

where

$$x = \ln X = \ln \frac{Y}{Y^n}, \tag{207}$$

where Y^n is natural output flow, i.e. output flow in the economy with no price adjustment costs ($\theta = 0$)

Dynamic Investment Savings Equation

- We can show that (see Assignment 2)

$$i_t = r_t^n + \pi_t + \mu_{X,t} - \sigma_{X,t}^2 = r_t^n + \pi_t + \mu_{x,t} - \frac{1}{2}\sigma_{x,t}^2, \tag{208}$$

where

$$dx_t = \mu_{x,t}dt + \sigma_x dZ_{x,t},\ dZ_{x,t}dZ_t = \rho_{X,t}dt \tag{209}$$

- It follows that

$$dx_t = (i_t - r^n - \pi_t + \frac{1}{2}\sigma_{x,t}^2)dt + \sigma_x dZ_{x,t} \tag{210}$$

FBSDE

- forward sde for output gap

$$dx_t = (i_t - r^n - \pi_t + \frac{1}{2}\sigma^2_{x,t})dt + \sigma_x dZ_{x,t},\ x_0 \text{ given} \tag{211}$$

- backward sde for inflation

$$d\pi_t = \delta\pi_t dt - \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x_t} - 1\right)dt + \frac{\partial \pi_t}{\partial x_t} dZ_{x,t}\ \lim_{T\to\infty} E_t[e^{-\delta(T-t)}\pi_T] = 0 \tag{212}$$

Nominal Interest Rate Rules I

- Make nominal interest rate rule depend on current inflation and output gap

$$i_t = v(\pi_t, x_t) \tag{213}$$

- Obtain a system of stochastic differential equations to pin down outgap and inflation
- forward sde for output gap

$$dx_t = (v(\pi_t, x_t) - r^n - \pi_t + \frac{1}{2}\sigma^2_{x,t})dt + \sigma_x dZ_{x,t},\ x_0 \text{ given} \tag{214}$$

- backward sde for inflation

$$d\pi_t = \delta\pi_t dt - \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x_t} - 1\right)dt + \frac{\partial \pi_t}{\partial x_t} dZ_{x,t} \lim_{T\to\infty} E_t[e^{-\delta(T-t)}\pi_T] = 0 \tag{215}$$

Nominal Interest Rate Rules II

- Suppose that $v(\pi, x) = a + \phi_\pi \pi + \phi_x x$. Then

$$dx_t = (a + (\phi_\pi - 1)\pi + \phi_x x - r^n + \frac{1}{2}\sigma_{x,t}^2)dt + \sigma_x dZ_{x,t}, \; x_0 \text{ given} \tag{216}$$

$$d\pi_t = \delta \pi_t dt - \frac{\epsilon - 1}{\theta}\left(e^{(1+\varphi)x_t} - 1\right)dt + \frac{\partial \pi_t}{\partial x_t} dZ_{x,t} \; \lim_{T\to\infty} E_t[e^{-\delta(T-t)}\pi_T] = 0 \tag{217}$$

- The nominal interest rate rule does not pin down σ_x.
- Need to think about more than interest rates when setting policy. Need to think about output gap volatility, which can be determined via policies affecting risk prices/risk premia.

Summary

- Quantitative Easing
- Some Stochastic Control Theory!
- Derived Basic New Keynesian Model with Risk
- Still need to apply Basic New Keynesian Model with Risk

Next Steps

- Monetary Policy in Liquidity Traps
- Quantitative Easing